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HIDDEN VARIABLE VECTOR VALUED FRACTAL INTERPOLATION FUNCTIONS

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Abstract

We present a method of construction of vector valued bivariate fractal interpolation functions on random grids in $\mathbb{R}^2$. Examples and applications are also included.

Keywords: Fractal Interpolation; IFS; Fractal Image Coding; Fractals; Bivariate Fractal Interpolation Functions; Hidden Variable; Fractal Approximation; FIF; FIS; Bivariate Fractal Interpolation Surfaces.

1. INTRODUCTION

With the aid of fractal geometry, we can construct surfaces that are extremely complex using only a handful of mappings. Lately, applications of fractal surfaces have been found in several scientific areas (such as metallurgy, geology, chemistry, medical sciences, image processing, etc.), where there is need to approximate surfaces of natural objects. Fractal geometry seems to be one of the best tools one can use to approximate natural surfaces, due to the complexity of the latter.

Fractal interpolation functions (FIFs) were introduced by Barnsley.[1] He used iterated function systems (IFSs) consisted of affine mappings, whose attractor is the graph of a continuous function that interpolates given data points. Massopust[2] introduced the construction of fractal interpolation surfaces (FISs) using affine IFSs over triangular grids. Geronimo and Hardin[3] and later Zhao[4] generalized Massopust's construction to allow consideration of more general boundary data and domains and to allow free contractivity factors. Xie and Sun[5] used bivariate IFSs in order to construct bivariate FISs. They claimed that their construction leads to attractors that are graphs of continuous functions $f : [a, b] \times [c, d] \to \mathbb{R}$. Unfortunately this was not the case. So Dalla[6] corrected their method and gave conditions that the interpolation data must satisfy in order for the attractor to be the graph of a continuous function. The construction was generalized and studied by Bouboulis *et al.*[7] Chand and Kapoor[8] tried to construct hidden variable bivariate FISs, but they made analogous mistakes as Xie and Sun. In this paper, we give a method of construction of vector valued bivariate FIFs. We present the IFS that leads to an attractor that is the graph of a continuous function $\vec{f} = (f_1, f_2) : [a, b] \times [c, d] \to \mathbb{R}^2$. The hidden variable bivariate FIS of Chand and Kapoor is a special case of our construction.

In Massopust,[2] we present an application of our method in image compression. The proposed algorithm assumes that the image is a surface and finds the parameters needed to construct a bivariate FIS to approximate it. One need to store only those parameters to memory. The performance of the algorithm is somewhat better than the well-known JPEG format.

2. CONSTRUCTION OF VECTOR VALUED FRACTAL INTERPOLATION FUNCTIONS

Consider the set $\{(x_n, y_m) = \vec{x}_{n,m} : n = 0, \dots, N, m = 0, \dots, M\} \subseteq [a, b] \times [c, d] = B$, where $a = x_0 < x_1 < \cdots < x_N = b$, $c = y_0 < y_1 < \cdots < y_M = d$ and the data $\Delta = \{(x_n, y_m, z_{n,m}, t_{n,m}) = (\vec{x}_{n,m}, \vec{z}_{n,m})\} \subseteq B \times \mathbb{R}^2 \subseteq \mathbb{R}^4$. Define $I_n = [x_{n-1}, x_n]$ and $J_m = [y_{m-1}, y_m]$ for $n = 1, 2, \dots, N$ and $m = 1, 2, \dots, M$. The aim is to construct a continuous vector valued function $\vec{f} = (f_1, f_2) : B \to \mathbb{R}^2$ such that $\vec{f}(\vec{x}_{n,m}) = \vec{z}_{n,m}$, $n = 0, \dots, N$, $m = 0, \dots, M$ (i.e. $\vec{f}$ be an interpolation function for the data Δ) and such that its graph be the attractor of an iterated function system. We define $\vec{w}_{n,m} : [a, b] \times [c, d] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4$:

$$\vec{w}_{n,m}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a_n x + b_n \\ c_m y + d_m \\ e_{n,m}x + f_{n,m}y + g_{n,m}xy + s_{n,m}z + s'_{n,m}t + k_{n.m} \\ \tilde{e}_{n,m}x + \tilde{f}_{n,m}y + \tilde{g}_{n,m}xy + \tilde{s}_{n,m}z + \tilde{s}'_{n,m}t + \tilde{k}_{n,m} \end{pmatrix}$$

$$= \begin{pmatrix} \phi_n(x) \\ \psi_m(y) \\ \Phi_{n,m}(x, y) + s_{n,m}z + s'_{n,m}t \\ \tilde{\Phi}_{n,m}(x, y) + \tilde{s}_{n,m}z + \tilde{s}'_{n,m}t \end{pmatrix} + \vec{c}_{n,m}.$$

We use the notation

$$\vec{w}_{n,m}\begin{pmatrix} \vec{x} \\ \vec{z} \end{pmatrix} = A_{n,m}\begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix} + \vec{c}_{n,m} + \begin{pmatrix} \vec{0} \\ \vec{\Phi}(\vec{x}) \end{pmatrix} + S_{n,m}\begin{pmatrix} \vec{0} \\ \vec{z} \end{pmatrix}, \tag{1}$$

where

$$A_{n,m} = \begin{pmatrix} a_n & 0 & 0 & 0 \\ 0 & c_m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \vec{c}_{n,m} = \begin{pmatrix} b_n \\ d_m \\ k_{n,m} \\ \tilde{k}_{n,m} \end{pmatrix},$$

$$\vec{\Phi}_{n,m}(\vec{x}) = \begin{pmatrix} \Phi_{n,m}(x, y) \\ \tilde{\Phi}_{n,m}(x, y) \end{pmatrix} = \begin{pmatrix} e_{n,m}x + f_{n,m}y + g_{n,m}xy \\ \tilde{e}_{n,m}x + \tilde{f}_{n,m}y + \tilde{g}_{n,m}xy \end{pmatrix}$$

and

$$S_{n,m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s_{n,m} & s'_{n,m} \\ 0 & 0 & \tilde{s}_{n,m} & \tilde{s}'_{n,m} \end{pmatrix}.$$

The constants of $S_{n,m}$ are fixed. The remaining constants that appear in $A_{n,m}, \vec{c}_{n,m}, \vec{\Phi}_{n,m}$ are defined by the equations

$$\begin{aligned} \vec{w}_{n,m}\begin{pmatrix} \vec{x}_{0,0} \\ \vec{z}_{0,0} \end{pmatrix} &= \begin{pmatrix} \vec{x}_{n-1,m-1} \\ \vec{z}_{n-1,m-1} \end{pmatrix} \\ \vec{w}_{n,m}\begin{pmatrix} \vec{x}_{N,0} \\ \vec{z}_{N,0} \end{pmatrix} &= \begin{pmatrix} \vec{x}_{n,m-1} \\ \vec{z}_{n,m-1} \end{pmatrix} \\ \vec{w}_{n,m}\begin{pmatrix} \vec{x}_{0,M} \\ \vec{z}_{0,M} \end{pmatrix} &= \begin{pmatrix} \vec{x}_{n-1,m} \\ \vec{z}_{n-1,m} \end{pmatrix} \\ \vec{w}_{n,m}\begin{pmatrix} \vec{x}_{N,M} \\ \vec{z}_{N,M} \end{pmatrix} &= \begin{pmatrix} \vec{x}_{n,m} \\ \vec{z}_{n,m} \end{pmatrix} \end{aligned} \tag{2}$$

for $n = 1, \ldots, N$, $m = 1, \ldots, M$ and they depend on the data Δ and $S_{n,m}$ (where $\vec{x} = \binom{x}{y}$, $\vec{z} = \binom{z}{t}$). The functions $\vec{w}_{n,m}$ are continuous on $B \times \mathbb{R}^2$ but not necessarily contractions.

Lemma 1. *Let $\vec{w} : B \times \mathbb{R}^2 \to B \times \mathbb{R}^2$ of the form*

$$\vec{w}\begin{pmatrix} \vec{x} \\ \vec{z} \end{pmatrix} = A\begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix} + \begin{pmatrix} \vec{0} \\ \vec{\Phi}(\vec{x}) \end{pmatrix} + S\begin{pmatrix} \vec{0} \\ \vec{z} \end{pmatrix}$$

where

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s & s' \\ 0 & 0 & \tilde{s} & \tilde{s}' \end{pmatrix}$$

with $\|A\|_1 = \max(|a|, |c|) < 1$, $\rho(S) < 1$ ($\rho(S)$ is the spectral radius of S), $\vec{\Phi} = (\Phi_1, \Phi_2) : B \to \mathbb{R}^2$ function with continuous partial derivatives on B. Then there exists a metric τ on $B \times \mathbb{R}^2$ equivalent to the euclidean norm $\|\cdot\|_2$ such that $\vec{w}$ be a contraction (the metric τ depends on A, Φ and S).

Proof. Let $S_0 = \begin{pmatrix} s & s' \\ \tilde{s} & \tilde{s}' \end{pmatrix}$, then $\rho(S_0) = \rho(S) < 1$. Choose $\epsilon > 0$ such that $\rho(S_0) + \epsilon < 1$. There exists a norm $\|\cdot\|_\epsilon$ in $\mathbb{R}^2$ such that

$$\|S_0\|_\epsilon \leq \rho(S_0) + \epsilon \tag{3}$$

(see Householder,[9] pp. 45–46). From the mean value theorem, the Cauchy-Scharz inequality and the fact that Φ_i have continuous partial derivatives on the compact set B, we conclude that

$$\begin{aligned} |\Phi_i(\vec{x}) - \Phi_i(\vec{x}')| &= |\nabla\Phi_i(\vec{\xi}_i)(\vec{x} - \vec{x}')| \\ &\leq \|\nabla\Phi_i(\vec{\xi}_i)\|_2\|\vec{x} - \vec{x}'\|_2 \\ &= M_i'\|\vec{x} - \vec{x}'\|_2, \quad i = 1, 2. \end{aligned}$$

As the norms in $\mathbb{R}^2$ are equivalent, we have that

$$\|\vec{\Phi}(\vec{x}) - \vec{\Phi}'(\vec{x})\|_\epsilon \leq M\|\vec{x} - \vec{x}'\|_1 \tag{4}$$

for some constant $M = M(\vec{\Phi}, S)$. As $\|A\|_1 < 1$, we may select $\theta = \theta(A, \vec{\Phi}, S) > 0$ with

$$\|A\|_1 + M\theta < 1. \tag{5}$$

We define the metric τ on $B \times \mathbb{R}^2$ by

$$\tau((\vec{x}, \vec{z}), (\vec{x}', \vec{z}')) = \|\vec{x} - \vec{x}'\|_1 + \theta\|\vec{z} - \vec{z}'\|_\epsilon.$$

Then for $(\vec{x}, \vec{z}), (\vec{x}', \vec{z}') \in B \times \mathbb{R}^2$, from (4), we have that

$$\begin{aligned} &\tau(\vec{w}(\vec{x}, \vec{z}), \vec{w}(\vec{x}', \vec{z}')) \\ &\quad \leq \|A\|_1\|\vec{x} - \vec{x}'\|_1 + \theta\|(\vec{\Phi}(\vec{x}) + S_0\vec{z}) \\ &\qquad - (\vec{\Phi}(\vec{x}') + S_0\vec{z}')\|_\epsilon \\ &\quad \leq \|A\|_1\|\vec{x} - \vec{x}'\|_1 + \theta(\|\vec{\Phi}(\vec{x}) - \vec{\Phi}(\vec{x}')\|_\epsilon \\ &\qquad + \|S_0\|_\epsilon\|\vec{z} - \vec{z}'\|_\epsilon) \\ &\quad \leq (\|A\|_1 + \theta M)\|\vec{x} - \vec{x}'\|_1 \\ &\qquad + \|S_0\|_\epsilon\theta\|\vec{z} - \vec{z}'\|_\epsilon \\ &\quad \leq \max\{\|A\|_1 + \theta M, \|S_0\|_\epsilon\} \\ &\qquad \cdot \tau((\vec{x}, \vec{z}), (\vec{z}', \vec{z}')). \end{aligned}$$

By (3) and (5), we have $s = \max\{\|A\|_1 + \theta M, \|S_0\|_\epsilon\} < 1$, hence $\vec{w}$ is a contraction on $\langle B \times \mathbb{R}^2, \tau\rangle$. □

Proposition 1. *Let the iterated vector valued function system*

$$\{B \times \mathbb{R}^2; \vec{w}_{n,m} : n = 1, \ldots, N, m = 1, \ldots, M\}$$

where $\vec{w}_{n,m}$ are defined from (1), *satisfy the conditions* (2) *and* $\max\{\rho(S_{n,m}) : n = 1, \ldots, N, m = 1, \ldots, M\} < 1$. *Then there exists a unique, nonempty, compact set $G \subseteq B \times \mathbb{R}^2$ such that $G = \bigcup_{n=1}^{N}\bigcup_{m=1}^{M}\vec{w}_{n,m}(G)$ and $\Delta \subseteq G$.*

Proof. By the conditions (2), we have that $\|A_{n,m}\|_1 < 1$. Hence $w_{n,m}$ is a contraction map on $\langle B \times \mathbb{R}^2, \tau\rangle$ by Lemma 1. Hence by the Banach fixed point theorem and well known results, we have the existence of G. □

The attractor G of $\{B \times \mathbb{R}^2; \vec{w}_{n,m} : n = 1, \ldots, N, m = 1, \ldots, M\}$ is a compact subset of

$B \times \mathbb{R}^2$ with $\Delta \subseteq G$ and it is not (in general) the graph of an interpolation function $\vec{f} = (f_1, f_2) : B \to \mathbb{R}^2$. We will give some conditions on the data Δ under which the set G will become the graph of a unique continuous fractal interpolation vector function.

Proposition 2. *Let $\Delta = \{(\vec{x}_{n,m}, \vec{z}_{n,m}) : n = 0, \ldots, N, m = 0, \ldots, M\}$ be a given data set such that if $x = x_N$ and $(1-\lambda)y_0 + \lambda y_M = y_m$, $\lambda \in [0,1]$, then $(1-\lambda)\vec{z}_{N,0} + \lambda \vec{z}_{N,M} = \vec{z}_{N,m}$ (analogous relations hold for $x = x_0$, $y = y_M$ and $y = y_0$). Let G be the attractor of $\{B \times \mathbb{R}^2; \vec{w}_{n,m} : n = 1, \ldots, N, m = 1, \ldots, M\}$ constructed under the conditions given in Proposition 1. Then there exists a continuous vector valued function $\vec{f} = (f_1, f_2) : B \to \mathbb{R}^2$ that interpolates the given data set Δ and its graph $G_{\vec{f}} = \{(\vec{x}, \vec{f}(\vec{x})) : \vec{x} \in B\} = G$.*

Proof. Let $\mathcal{F}$ be the set of continuous functions $\vec{f} = (f_1, f_2)$ defined on B such that

$$\begin{aligned}
&\vec{f}(x_0, (1-\lambda)y_0 + \lambda y_M) \\
&\quad = ((1-\lambda)z_{0,0} + \lambda z_{0,M}, (1-\lambda)t_{0,0} + \lambda t_{0,M}) \\
&\vec{f}(x_N, (1-\lambda)y_0 + \lambda y_M) \\
&\quad = ((1-\lambda)z_{N,0} + \lambda z_{N,M}, (1-\lambda)t_{N,0} + \lambda t_{N,M}) \\
&\vec{f}((1-\lambda)x_0 + \lambda x_N, y_0) \\
&\quad = ((1-\lambda)z_{0,0} + \lambda z_{N,0}, (1-\lambda)t_{0,0} + \lambda t_{N,0}) \\
&\vec{f}((1-\lambda)x_0 + \lambda x_N, y_M) \\
&\quad = ((1-\lambda)z_{0,M} + \lambda z_{N,M}, (1-\lambda)t_{0,M} + \lambda t_{N,M})
\end{aligned}$$

(for $\lambda \in [0,1]$), i.e. $f_1, f_2 : B \to \mathbb{R}$ are linear on the edges of $B = [a,b] \times [c,d]$. Then $\mathcal{F}$ is a closed subset of the Banach space $\langle C_{\mathbb{R}^2}(B), \|\cdot\|_\infty \rangle$ (where $\|\cdot\|_\infty$ the sup-norm). Hence $\mathcal{F}$ is a complete matric space. We introduce the Read-Bajraktarevic operator $T : \mathcal{F} \to \mathcal{F}$ with

$$\begin{aligned}
T\vec{f}(x,y) = {} & \vec{\Phi}_{n,m}(\phi_n^{-1}(x), \psi_m^{-1}(y)) \\
& + S_{n,m}\vec{f}(\phi_n^{-1}(x), \psi_m^{-1}(y)) + \vec{K}_{n,m}
\end{aligned}$$

where $(x,y) \in I_n \times J_m = [x_{n-1}, x_n] \times [y_{m-1}, y_m]$. Here, for simplicity, we set

$$S_{n,m} = \begin{pmatrix} s_{n,m} & s'_{n,m} \\ \tilde{s}_{n,m} & \tilde{s}'_{n,m} \end{pmatrix} \quad \text{and}$$

$$\vec{K}_{n,m} = \begin{pmatrix} k_{n,m} \\ \tilde{k}_{n,m} \end{pmatrix}.$$

We have to prove that $T\vec{f}(x,y)$ is well defined. If (x,y) lies on one of the edges of $I_n \times J_m$, then it lies on one of the edges of $I_{n+1} \times J_m$, $I_n \times J_{m+1}$, $I_{n-1} \times J_m$, $I_n \times J_{m-1}$. We have to prove that the value $T\vec{f}(x,y)$ of such (x,y) is the same regardless of the set ($I_n \times J_m$, $I_{n+1} \times J_m$, $I_n \times J_{m+1}$, $I_{n-1} \times J_m$, $I_n \times J_{m-1}$) to which it belongs. Let $(x,y) = (x_n, (1-\lambda)y_{m-1} + \lambda y_m) \in [0,1]$ be a point on the common edge $[(x_n, y_{m-1}), (x_n, y_m)]$ of $I_n \times J_m$ and $I_{n+1} \times J_m$. Then, if we consider $(x,y) \in I_n \times J_m$ we have that

$$\begin{aligned}
&T\vec{f}(x_n, (1-\lambda)y_{m-1} + \lambda y_m) \\
&\quad = \vec{\Phi}_{n,m}(x_N, (1-\lambda)y_0 + \lambda y_M) \\
&\qquad + S_{n,m}\vec{f}(x_N, (1-\lambda)y_0 + \lambda y_M) + \vec{K}_{n,m} \\
&\quad = (1-\lambda)\vec{\Phi}_{n,m}(x_N, y_0) + \lambda \vec{\Phi}_{n,m}(x_N, y_M) \\
&\qquad + (1-\lambda)S_{n,m}\vec{f}(x_N, y_0) \\
&\qquad + \lambda S_{n,m}\vec{f}(x_n, y_m) + \vec{K}_{n,m} \\
&\qquad \text{(since } \vec{f} \in \mathcal{F} \text{ and } \vec{\Phi}_{n,m} \text{ is linear)} \\
&\qquad\ \text{in each variable)} \\
&\quad = (1-\lambda)(\vec{\Phi}_{n,m}(x_N, y_0) \\
&\qquad + S_{n,m}\vec{f}(x_N, y_0) + \vec{K}_{n,m}) \\
&\qquad + \lambda(\vec{\Phi}_{n,m}(x_N, y_M) + S_{n,m}\vec{f}(x_N, y_m) + \vec{K}_{n,m}) \\
&\quad = (1-\lambda)\vec{z}_{n,m-1} + \lambda \vec{z}_{n,m} \quad \text{[by conditions (2)].}
\end{aligned}$$

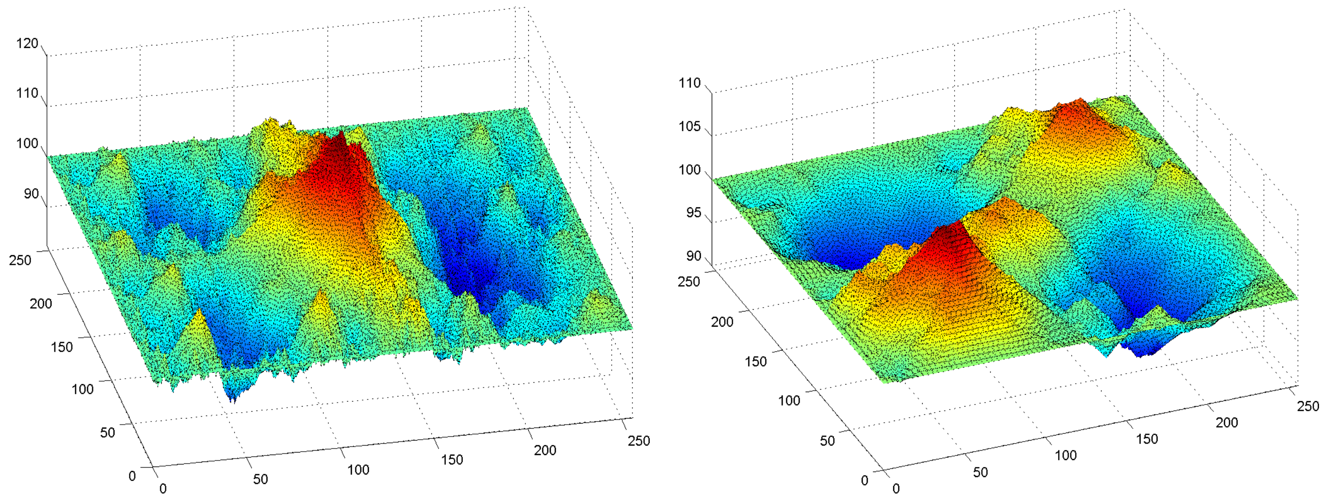
Thus,

$$\begin{aligned}
&T\vec{f}(x_n, (1-\lambda)y_{m-1} + \lambda y_m) \\
&\quad = (1-\lambda)\vec{z}_{n,m-1} + \lambda \vec{z}_{n,m}.
\end{aligned} \tag{6}$$

Next, if we consider $(x,y) \in I_{n+1} \times J_m$ we can find that $T\vec{f}(x,y) = (1-\lambda)\vec{z}_{n,m-1} + \lambda \vec{z}_{n,m}$. Hence $T\vec{f}$ is well defined on B and by its definition it is continuous on B and interpolates the data Δ [take $\lambda = 0$ or $\lambda = 1$ in (6)]. Finally, it remains to prove that $T\vec{f} \in \mathcal{F}$ (i.e. $T\vec{f}$ is linear for $x = x_0, x_N$ and $y = y_0, y_M$). Consider the point $(x_N, (1-\lambda)y_0 + \lambda y_M)$ ($\lambda \in [0,1]$) then there are $m \in \{1, 2, \ldots, M\}$ and $\lambda' \in [0,1]$ such that $(x_N, (1-\lambda)y_0 + \lambda y_M) = (x_N, (1-\lambda')y_{m-1} + \lambda' y_m) \in I_N \times J_m$. As in (6), we conclude that

$$\begin{aligned}
&T\vec{f}(x_N, (1-\lambda)y_0 + \lambda y_M) \\
&\quad = T\vec{f}(x_N, (1-\lambda')y_{m-1}, \lambda' y_m) \\
&\quad = (1-\lambda')\vec{z}_{N,m-1} + \lambda' \vec{z}_{N,m} \\
&\quad = (1-\lambda)\vec{z}_{N,0} + \lambda z_{N,M} \\
&\qquad \text{(see the condition of the proposition).}
\end{aligned}$$

Hence $Tf \in \mathcal{F}$. Proceeding as in Dalla[6] we can prove that $G_{\vec{f}} = \tilde{G}$. □



(a) Projection on x-y-z space.

(b) Projection on x-y-t space.

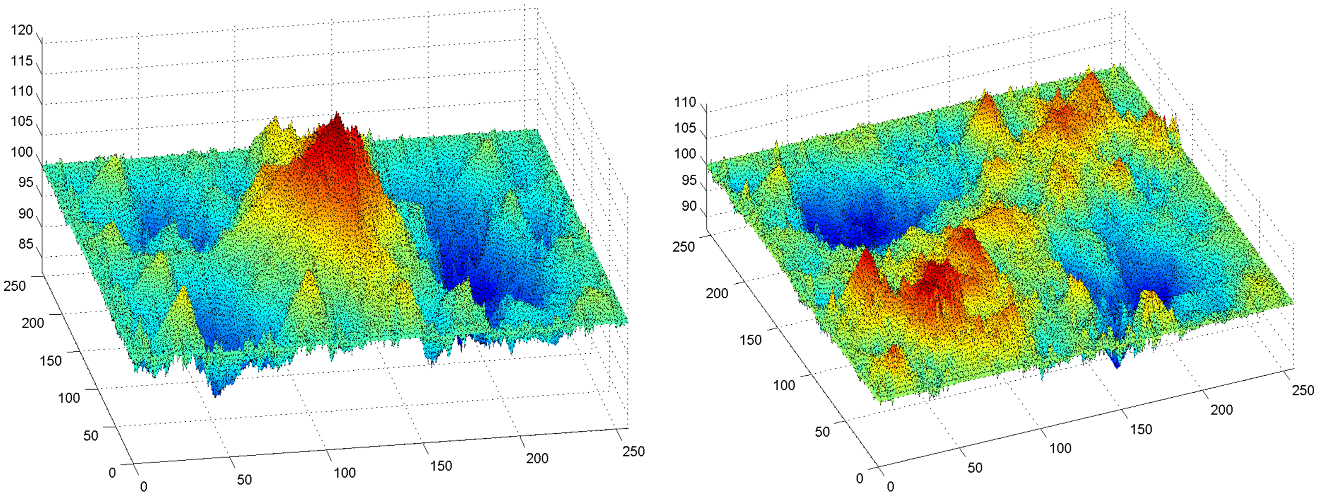
Fig. 1 In this example we use the data of Table 1. Here $\tilde{s}_{i,j} = 0$.[8]

Table 1 Interpolation points (data) and contraction factors $(s, s', \tilde{s}, \tilde{s}')$.

Data Points				Data Points				Data Points			
x	**y**	**t**	**z**	**x**	**y**	**t**	**z**	**x**	**y**	**t**	**z**
0	0	100	100	0	128	100	100	0	256	100	100
64	0	100	100	64	128	98	110	64	256	100	100
128	0	100	100	128	128	105	120	128	256	100	100
192	0	100	100	192	128	95	110	192	256	100	100
256	0	100	100	256	128	100	100	256	256	100	100
0	64	100	100	0	192	100	100				
64	64	110	90	64	192	90	90				
128	64	100	100	128	192	90	85				
192	64	90	90	192	192	95	90				
256	64	100	100	256	192	107	100				

s_{ij}					s'_{ij}				
	j					**j**			
i	**1**	**2**	**3**	**4**	**i**	**1**	**2**	**3**	**4**
1	0.45	0.35	−0.3	0.35	**1**	0.2	0.1	0.2	0.1
2	0.4	−0.15	0.25	−0.35	**2**	0.1	0.3	−0.1	0.1
3	−0.45	0.15	−0.45	0.2	**3**	0.2	−0.4	0.1	0.3
4	−0.15	−0.25	0.45	0.25	**4**	−0.3	0.2	−0.1	0.1

$\tilde{s}_{ij}$					$\tilde{s}'_{ij}$				
	j					**j**			
i	**1**	**2**	**3**	**4**	**i**	**1**	**2**	**3**	**4**
1	0	0	0	0	**1**	−0.2	0.3	0.1	0.2
2	0	0	0	0	**2**	0.1	0.3	−0.1	0.1
3	0	0	0	0	**3**	−0.3	−0.3	−0.1	0.3
4	0	0	0	0	**4**	−0.1	0.1	−0.3	0.2



(a) Projection on x-y-z space.

(b) Projection on x-y-t space.

Fig. 2 General case. We use the data points of the previous example and of Table 2.

Table 2 Additional contraction factors.

	$\tilde{s}_{ij}$			
	j			
i	**1**	**2**	**3**	**4**
1	−0.2	0.3	0.1	0.2
2	0.1	0.3	−0.1	0.1
3	−0.3	−0.3	−0.1	0.3
4	−0.1	0.1	−0.3	0.2

3. EXAMPLES

In Figs. 1 and 2, one can see some vector valued FISs constructed using our general construction method. Of course, there is no other way to actually "see" the surfaces since they are vector valued. The figures show the "projections" on x-y-z and x-y-t spaces. The method used for constructing the figures presented here is based on the deterministic iteration algorithm (DIA).

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