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Construction of orthogonal multi-wavelets using generalized-affine fractal interpolation functions

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We present a new construction of fractal interpolation surfaces defined on arbitrary rectangular lattices. We use this construction to form finite sets of fractal interpolation functions (FIFs) that generate multiresolution analyses of $L_2(\mathbb{R}^2)$ of multiplicity r. These multiresolution analyses are based on the dilation properties of the construction. The associated multi-wavelets are orthogonal and discontinuous functions. We give concrete examples to illustrate the method and generalize it to form multiresolution analyses of $L_2(\mathbb{R}^d)$, $d > 2$. To this end, we prove some results concerning the Hölder exponent of FIFs defined on $[0, 1]^d$.

Keywords: fractal interpolation functions; fractal interpolation surfaces; fractals; moments; Hölder; multi-wavelets.

1. Introduction

Fractal interpolation, as introduced by Barnsley (1986) (see also Barnsley *et al.*, 1989), is an alternative to traditional interpolation techniques, which gives a broader set of interpolants. In fact, many traditional interpolation techniques (splines, hermite polynomials, etc.) are included as special cases. Its main differences consist (a) in the definition of a functional relation (see (11)) that implies a self-similarity in small scales, (b) in the constructive way (through iterations), that it is used to compute the interpolant, instead of the descriptive one (usually a formula) provided by the classical methods and (c) in the usage of some parameters, which are usually called vertical scaling factors, that are strongly related with the fractal dimension of the interpolant. It was these properties (and especially the second one) that led Geronimo, Hardin, Kessler and Massopust to use fractal interpolation functions (FIFs) for the generation of multi-wavelets (see Hardin *et al.*, 1992; Geronimo *et al.*, 1994) before the general concept of multiresolution analysis of multiplicity r had been introduced in Goodman *et al.* (1993) and Goodman & Lee (1994) (the construction presented in Geronimo *et al.* (1994) is known as Geronimo–Hardin–Massopust multi-wavelets and was latter constructed by Chui & Lian (1996) without using fractal interpolation). Their work led to the celebrated Donovan–Geronimo–Hardin–Massopust orthogonal multi-wavelets (see Donovan *et al.*, 1996). The present work is highly motivated by their results (and especially from Hardin *et al.*, 1992). We must point out that the construction of multi-wavelets via fractal interpolation differs in a lot of ways from the standard wavelet techniques. The usual approach is to seek for a solution of the refinement equation satisfying several properties (such as orthogonality, continuity, high approximation order, etc.). The fractal interpolation approach on the other hand makes use of suitable FIFs constructed to incorporate the desired properties. Interesting works regarding the

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construction of wavelets via fractal interpolation can also be found in Hardin & Marasovich, 1999; Kessler, 2007.

All the aforementioned constructions were based on FIFs defined on compact subsets of $\mathbb{R}$. There have been some efforts to generate multiresolution analyses from fractal interpolation surfaces (FISs), especially on triangulations (see Geronimo & Hardin, 1993; Kessler, 2000) following the work of Massopust (1990). Of course, one can always take the tensor product of the 1D case, but working directly on two dimensions has certain advantages. Our work is based on a new construction of fractal interpolation on rectangular lattices, which can be generalized to produce FIFs defined on $[0, 1]^d$. The structure of the paper is as follows: In Section 2, we briefly review the concept of iterated function systems (IFSs) and fractal interpolation as laid out by Barnsley for the 1D case. In Section 3, we introduce the new construction of fractal interpolation on rectangular lattices of $[0, 1]^2$. Section 4 deals with the computation of the inner product of two FISs, which is necessary for the computation of the multi-wavelets. In Section 5, we prove some results regarding the scaling properties of the construction. Section 6 deals with the generation of the multiresolution analyses and the corresponding multi-wavelets. Finally, in Section 7 we generalize the construction to $[0, 1]^d$ and generate multiresolution analyses of $L_2(\mathbb{R}^d)$. The result regarding the Hölder exponent of the constructed FIF is also found there since it is essential for the validation of the construction.

2. Background

In this section, we briefly review the concept of fractal interpolation, as given by Barnsley for the 1D case.

2.1 *Iterated function system–recurrent iterated function system*

Perhaps, the most typical way to construct fractal sets is via an IFS. An IFS $\{X; w_{1-N}\}$ is defined as a pair consisting of a complete metric space (X, ρ), together with a finite set of continuous, contractive mappings $w_i \colon X \to X$, with respective contraction factors s_i, for $i = 1, 2, \ldots, N$ $(N \geqslant 2)$. The attractor of an IFS is the unique set E, for which $E = \lim_{k\to\infty} W^k(A_0)$ for every starting compact set A_0, where

$$W(A) = \bigcup_{i=1}^{N} w_i(A) \quad \text{for all } A \in \mathcal{H}(X)$$

and $\mathcal{H}(X)$ is the complete metric space of all non-empty compact subsets of X with respect to the Hausdorff metric h (for the definition of the Hausdorff metric, properties of $\langle \mathcal{H}(X), h\rangle$ and examples of IFS, see Barnsley & Demko (1985) and Barnsley (1993) among others).

A notion closely related with IFS is that of the ‘recurrent iterated function system’ (RIFS) that allows the construction of even more complicated sets. However, in this paper we will not deal with RIFS, therefore we omit its definition.

2.2 *Fractal interpolation functions*

Barnsley (1986) was the first who considered the possibility of using IFS for data interpolation. He constructed functions that interpolate arbitrary data points, whose graphs are attractors of specific IFSs or RIFSs (see Barnsley, 1986; Barnsley *et al.*, 1989). Barnsley called those functions ‘fractal interpolation functions’ (FIFs) due to the fact that they may have non-integer fractal dimension. Here, we briefly describe this construction based on IFSs.

Let $X = [0,1] \times \mathbb{R}$ and $\Delta = \{(x_i, y_i)\colon i = 0, 1, \ldots, N\}$ be an interpolation set with $N+1$ interpolation points such that $0 = x_0 < x_1 < \cdots < x_N = 1$. The interpolation points divide [0, 1] into N intervals $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, N$, which we call 'domains'. We define $\delta_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$

Next, we define N mappings of the form

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} T_i(x) \\ F_i(x,y) \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, N, \tag{1}$$

where $T_i(x) = a_i x + b_i$ and $F_i(x, y) = s_i y + p_i(x)$ ($p_i(x)$ is a polynomial). Each map w_i is constrained to map the end points of the region [0, 1] to the end points of the domain I_i. That is,

$$w_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix}, \quad w_i \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, N. \tag{2}$$

Vertical segments are mapped to vertical segments scaled by the factor s_i. The parameter s_i is called the 'vertical scaling factor' of the map w_i.

It is easy to show that if $|s_i| < 1$, then there is a metric d equivalent to the Euclidean metric such that w_i is a contraction (i.e. there is $\hat{s}_i$: $0 \leqslant \hat{s}_i < 1$ such that $d(w_i(\vec{x}), w_i(\vec{y})) \leqslant \hat{s}_i d(\vec{x}, \vec{y})$; see Barnsley, 1993).

Finally, we consider $\langle C([x_0, x_N]), \|\cdot\|_\infty\rangle$, where $\|\phi\|_\infty = \max\{|\phi(x)|, x \in [x_0, x_N]\}$ and the complete metric subspace $\mathbb{F}_{\Delta,\boldsymbol{s}} = \{g \in C([x_0, x_N])\colon F_i(x_0, g(x_0)) = y_{i-1}, F_i(x_N, g(x_N)) = y_i$ for $i = 1, 2, \ldots, N\}$. The Read–Bajraktarevic operator $T_{\Delta,\boldsymbol{s}}\colon \mathbb{F}_{\Delta,\boldsymbol{s}} \to \mathbb{F}_{\Delta,\boldsymbol{s}}$ is defined as follows:

$$(T_{\Delta,\boldsymbol{s}} g)(x) = F_i(T_i^{-1}(x), g(T_i^{-1}(x))), \quad \text{for } x \in [x_{i-1}, x_i], \ \ i = 1, 2, \ldots, N,$$

where $\boldsymbol{s} = (s_1, \ldots, s_N)^\top$. It is easy to verify that $T_{\Delta,\boldsymbol{s}} g$ is well defined and that $T_{\Delta,\boldsymbol{s}}$ is a contraction with respect to the ρ_∞ metric. According to the Banach fixed-point theorem, there exists a unique $f \in \mathbb{F}_\Delta$ such that $T_{\Delta,\boldsymbol{s}} f = f$. If f_0 is any interpolation function and $f_n = T^n_{\Delta,\boldsymbol{s}} f_0$, where $T^n_{\Delta,\boldsymbol{s}} = T_{\Delta,\boldsymbol{s}} \circ T_{\Delta,\boldsymbol{s}} \circ \cdots \circ T_{\Delta,\boldsymbol{s}}$, then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f. The graph of the function f is the attractor of the IFS $\{X, w_{1-N}\}$ associated with the interpolation points (see Barnsley, 1993). Note that f interpolates the points of Δ for any selection of the parameters of the polynomials p_i that satisfies (2). We will refer to a function of this nature as an FIF. It is readily proved by the above that the FIF is the unique function f that satisfies the functional relation

$$(T_{\Delta,\boldsymbol{s}} f)(x) = F_i(T_i^{-1}(x), f(T_i^{-1}(x))). \tag{3}$$

Likewise, f is the unique function whose graph G satisfies the relation

$$G = \bigcup_{i=1}^{N} w_i(G). \tag{4}$$

Let us consider the case where w_i are affine:

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_i(x,y) \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & s_i \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_i \\ d_i \end{pmatrix}, \quad \text{for } i = 1, 2, \ldots, N. \tag{5}$$

Here, $p_i(x) = c_i x + f_i$. The FIF that corresponds to the above IFS is called 'affine' FIF.

From (2) four linear equations arise, which can always be solved for a_i, c_i, b_i, d_i in terms of the coordinates of the interpolation points and the vertical scaling factor s_i. Thus, once the contraction factor s_i for each map has been chosen, the remaining parameters may be easily computed (see Barnsley, 1993).

3. Construction of FISs

Many authors tried to generalize Barnsley's construction on $\mathbb{R}^3$ to produce FISs. More credited are the works of Massopust, who was the first to consider the problem and also wrote a book on the subject (see Massopust, 1990, 1994), Bouboulis and Dalla (see Bouboulis *et al.*, 2006; Bouboulis & Dalla, 2007b,c; Dalla, 2002), Malysz (see Malysz, 2006), Zhao (see Zhao, 1996), Wang (see Wang, 2006) and Feng (see Feng, 2008). We should also mention the construction by Xie & Sun (1997) which leads to compact sets that interpolate data points on $\mathbb{R}^3$. However, in most of these attempts the construction uses either interpolation points, that are restricted to be collinear in the borders of $I = [0,1]^2$, or maps with equal vertical scaling factors. A general construction that can be applied to arbitrary data points on $\mathbb{R}^n$ was presented recently in Bouboulis & Dalla (2007a). The main difference of this approach is that it takes into account not only the values of the interpolation points but also the values of the borders of the rectangular grid, which are chosen *a priori*. The method presented here is an extension.

Consider a data set

$$\Delta = \{(x_i, y_j, z_{i,j}) \in I \times \mathbb{R};\, i = 0, 1, \ldots, N,\, j = 0, 1, \ldots, M\}$$

such that $0 = x_0 < x_1 < \cdots < x_N = 1$ and $0 = y_0 < y_1 < \cdots < y_M = 1$, $N, M \in \mathbb{N}$, where $I = [0,1]^2$, which contains in total $(N+1)\cdot(M+1)$ points. We also define the set

$$\Delta' = \{(x_i, y_j);\, i = 0, 1, \ldots, N,\, j = 0, 1, \ldots, M\}.$$

The points of Δ' divide $[0,1]^2$ into $N \cdot M$ regions

$$I_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j],$$

for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$.

Next, we consider $N \cdot M$ mappings of the form

$$W_{i,j}\colon [0,1]^2 \times \mathbb{R} \to I_{i,j} \times \mathbb{R}\colon W_{i,j}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}_{i,j}(x,y) \\ F_{i,j}(x,y,z) \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}_{i,j}(x,y) \\ s_{i,j} z + p_{i,j}(x,y) \end{pmatrix}, \tag{6}$$

with

$$\boldsymbol{T}_{i,j}(x,y) = \begin{pmatrix} T_{1,i}(x) \\ T_{2,j}(y) \end{pmatrix} = \begin{pmatrix} a_{1,i} x + b_{1,i} \\ a_{2,j} y + b_{2,j} \end{pmatrix},$$

for all $(x,y) \in [0,1]^2$, where $p_{i,j}$ is a continuous function on $[0,1]^2$ that satisfies a Hölder condition

$$|p_{i,j}(x', y') - p_{i,j}(x,y)| \leqslant L_{i,j} \|(x' - x, y' - y)\|_1^{h_{i,j}},$$

for some constants $L_{i,j} > 0$, $h_{i,j} > 0$ and $s_{i,j} \in (-1, 1)$, for all $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. We can use any other norm equivalent to the Euclidean one. The $\|\cdot\|_1$ norm was chosen to simplify the notation in the proof of Proposition 7.2. The parameters $s_{i,j}$ are called vertical scaling factors. We confine the map $W_{i,j}$ so that it maps the interpolation points that lie on the vertices of $[0, 1]^2$ to the interpolation points that lie on the vertices of $I_{i,j}$. Hence, we obtain the following relations:

$$T_{1,i}(x_0) = x_{i-1}, \quad T_{1,i}(x_N) = x_i,$$

$$T_{2,j}(y_0) = y_{j-1}, \quad T_{2,j}(y_N) = y_j$$

and

$$F_{i,j}(x_0, y_0) = z_{i-1,j-1}, \quad F_{i,j}(x_0, y_N) = z_{i-1,j},$$

$$F_{i_1,i_2}(x_N, y_0) = z_{i,j-1}, \quad F_{i,j}(x_N, y_M) = z_{i,j}.$$

It is easy to show that there exists a metric ρ_θ (equivalent with the Euclidean metric) such that $W_{i,j}$ is a contraction for all $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. To this end, consider the metric ρ_1 defined on $[0, 1]^2$ as follows:

$$\rho_1((x', y'), (x, y)) = \|(x' - x, y' - y)\|_1^h,$$

where $h = \min\left\{h_{i_1,i_2}\right\}$, and the metric

$$\rho_\theta((x', y', z'), (x, y, z)) = \rho_1((x', y'), (x, y)) + \theta|z' - z|$$

defined on $[0, 1]^2 \times \mathbb{R}$, where θ is properly specified (for a complete proof, see, e.g. Wang, 2006; Bouboulis & Dalla, 2007a). Therefore, the IFS $\{[0, 1]^2 \times \mathbb{R}, W_{i,j}, i = 1, 2, \ldots, N, j = 1, 2, \ldots, M\}$ has a unique attractor G. In general, G is a compact subset of $\mathbb{R}^3$ containing the points of Δ. The following proposition gives conditions so that G is the graph of a continuous function f. As mentioned above, these conditions involve points that lie on $\partial I_{i_1,i_2} \times \mathbb{R}$, for all $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$ (where $\partial I_{i_1,i_2}$ is the boundary of $I_{i,j}$). The proof can be found in Bouboulis & Dalla (2007a) (in the case of a Lipschitz condition, but it can be easily extended).

PROPOSITION 3.1 Let $h \in C([0, 1]^2)$ be a function that interpolates the points of Δ (i.e. $h(x_i, y_j) = z_{i,j}$) such that it satisfies a Hölder condition. If the IFS defined above satisfies the conditions

$$F_{i,j}(x_0, y, h(x_0, y)) = h(x_{i-1}, T_{2,j}(y)), \tag{7}$$

$$F_{i,j}(x_N, y, h(x_N, y)) = h(x_i, T_{2,j}(y)), \tag{8}$$

$$F_{i,j}(x, y_0, h(x, y_0)) = h(T_{1,i}(x), y_{j-1}), \tag{9}$$

$$F_{i,j}(x, y_M, h(x, y_M)) = h(T_{1,i}(x), y_j), \tag{10}$$

for all $(x, y) \in [0, 1]^2$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$, then its attractor G is the graph of a continuous function f that interpolates the data points. Moreover, f is the unique function that satisfies the functional relation

$$f(x, y) = F_{i,j}(\boldsymbol{T}_{i,j}^{-1}(x, y), f(\boldsymbol{T}_{i,j}^{-1}(x, y))), \tag{11}$$

for all $(x, y) \in I_{i,j}$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$.

As in the case of the 1D FIF, f is the unique function whose graph G satisfies

$$G = \bigcup_{i=1}^{N} \bigcup_{j=1}^{M} W_{i,j}(G). \tag{12}$$

The corresponding Read–Bajractarevic operator $\mathcal{T}_{\Delta,h,s}$ is defined as

$$\mathcal{T}_{\Delta,h,s}\colon \mathbb{F}_{\Delta,h,s} \to \mathbb{F}_{\Delta,h,s}\colon (\mathcal{T}_{\Delta,h,s}g)(x) = F_{i,j}(T_{i,j}^{-1}(x,y), g(T_{i,j}^{-1}(x,y))),$$

if $x \in I_{i,j}$, for $i = 1, \dots, N$, $j = 1, \dots, M$, where $\mathbb{F}_{\Delta,h,s}$ is the space of all continuous functions on $[0,1]^2$ that satisfy (7–10).

The relations (7–10) define a functional system that consists of $4 \cdot N \cdot M$ equations that associate $F_{i,j}$ with h (only at points of $\partial I_{i,j}$). Considering that $F_{i,j}(x,y,z) = s_{i,j}z + p_{i,j}(x,y)$, we obtain the system

$$p_{i,j}(x_0, y) = h(x_{i-1}, T_{2,j}(y)) - s_{i,j} \cdot h(x_0, y), \tag{13}$$

$$p_{i,j}(x_N, y) = h(x_i, T_{2,j}(y)) - s_{i,j} \cdot h(x_N, y), \tag{14}$$

$$p_{i,j}(x, y_{M-1}) = h(T_{1,i}(x), y_{j-1}) - s_{i,j} \cdot h(x, y_0), \tag{15}$$

$$p_{i,j}(x, y_M) = h(T_{1,i}(x), y_j) - s_{i,j} \cdot h(x, y_M), \tag{16}$$

for all $(x, y) \in [0,1]^2$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$, where $s_{i,j}$ are free parameters. In this paper, we limit our interest only to the case where

$$p_{i,j}(x,y) = r_{1,i,j}(y)x + r_{2,i,j}(x)y + q_{1,i,j}(y) + q_{2,i,j}(x), \tag{17}$$

for all $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$. Solving the system of equations, we obtain

$$r_{1,i,j}(y) = \frac{h(x_i, T_{2,j}(y)) - h(x_{i-1}, T_{2,j}(y))}{x_N - x_0} - s_{i,j}\frac{h(x_N, y) - h(x_0, y)}{x_N - x_0},$$

$$q_{1,i,j}(y) = h(x_{i-1}, T_{2,j}(y)) - s_{i,j}h(x_0, y) - r_{1,i,j}(y)x_0,$$

$$r_{2,i,j}(x) = \frac{h(T_{1,i}(x), y_j) - h(T_{1,i,j}(x), y_{j-1})}{y_M - y_0} - s_{i,j}\frac{h(x, y_M) - h(x, y_0)}{y_M - y_0}$$

$$- \frac{r_{1,i,j}(y_M) - r_{1,i,j}(y_0)}{y_M - y_0}x - \frac{q_{1,i,j}(y_M) - q_{1,i,j}(y_0)}{y_M - y_0},$$

$$q_{2,i,j}(x) = h(T_{1,i}(x), y_{j-1}) - s_{i,j}h(x, y_0) - r_{1,i,j}(y_{j-1})x - r_{2,i,j}(x)y_0 - q_{1,i,j}(y_0),$$

for all $x, y \in [0,1]^2$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$. Therefore, if one constructs $N + M$ 1D interpolants that satisfy a Hölder condition and interpolate the given data, then any IFS consisting of mappings of the above form satisfies the conditions of the propositions. More specifically, we consider the functions u_i that interpolate the points of $\varDelta_i = \{(x_i, y_j, z_{i,j}), j = 0, 1, \ldots, M\}$, for $i = 0, 1, \ldots, N$, and the functions υ_j that interpolate the points of $\tilde{\varDelta}_j = \{(x_i, y_j, z_{i,j}), i = 0, 1, \ldots, N\}$, for $j = 0, 1, \ldots, M$. Then (17) gives $p_{i,j}(x, y)$ in terms of $u_i(y)$ and $\upsilon_j(x)$, for $i = 1, \ldots, N$, $j = 1, \ldots, M$. In particular,

$$r_{1,i,j}(y) = \frac{u_i(T_{2,j}(y)) - u_{i-1}(T_{2,j}(y))}{x_N - x_0} - s_{i,j}\frac{u_N(y) - u_0(y)}{x_N - x_0}, \tag{18}$$

$$q_{1,i,j}(y) = u_{i-1}(T_{2,j}(y)) - s_{i,j} \cdot u_0(y) - r_{1,i,j}(y)x_0, \tag{19}$$

$$r_{2,i,j}(x) = \frac{\upsilon_j(T_{1,i}(x)) - \upsilon_{j-1}(T_{1,i}(x))}{y_M - y_0} - s_{i,j}\frac{\upsilon_M(x) - \upsilon_0(x)}{y_M - y_0}$$
$$- \frac{r_{1,i,j}(y_M) - r_{1,i,j}(y_0)}{y_M - y_0}x - \frac{q_{1,i,j}(y_M) - q_{1,i,j}(y_0)}{y_M - y_0}, \tag{20}$$

$$q_{2,i,j}(x) = \upsilon_{j-1}(T_{1,i}(x)) - s_{i,j} \cdot \upsilon_0(x) - r_{1,i,j}(x_0) - q_{1,i,j}(x_0), \tag{21}$$

for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Substituting in (17) and taking into account that in our case $x_0 = y_0 = 0$ and $x_N = y_M = 1$, we obtain

$$\begin{aligned} p_{i,j}(x, y) = {} & s_{i,j}(x-1)(y-1)z_{0,0} - s_{i,j}(x-1)yz_{0,M} - z_{i-1,j-1} + xz_{i-1,j-1} \\ & + yz_{i-1,j-1} - xyz_{i-1,j-1} - yz_{i-1,j} + xyz_{i-1,j} - xz_{i,j-1} + xyz_{i,j-1} \\ & - xyz_{i,j} + s_{i,j}xz_{N,0} - s_{i,j}xyz_{N,0} + s_{i,j}xyz_{N,M} - s_{i,j}u_0(y) + s_{i,j}xu_0(y) \\ & + u_{i-1}(T_{2,j}(y)) - xu_{i-1}(T_{2,j}(y)) + xu_i(T_{2,j}(y)) - s_{i,j}xu_N(y) - s_{i,j}\upsilon_0(x) \\ & + s_{i,j}y\upsilon_0(x) + \upsilon_{j-1}(T_{1,i}(x)) - y\upsilon_{j-1}(T_{1,i}(x)) + y\upsilon_j(T_{1,i}(x)) - s_{i,j}y\upsilon_M(x), \end{aligned} \tag{22}$$

for all $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. This IFS gives rise to an FIS. Figure 1 shows the graph of an FIS, where the 1D interpolants are polygonal lines. (More examples and a more detailed description using RIFS can be found in Bouboulis & Dalla, 2007a.)

In an attempt to make this construction to depend explicitly on the original interpolation points, one may consider that the 1D interpolants are affine FIFs constructed as mentioned in Section 2.2 (affine FIFs satisfy a Hölder condition; see Massopust (1994) or Section 7 for a more general result). In this case, u_i are the affine FIFs associated with the set $\varDelta_i$, together with some arbitrary vertical scaling factors $\sigma_{i,j}$, $j = 1, \ldots, M$, for $i = 0, 1, \ldots, N$. Similarly, υ_j is the affine FIF associated with the set $\tilde{\varDelta}_j$, together with vertical scaling factors $\tilde{\sigma}_{i,j}$, $i = 1, \ldots, N$, for $j = 0, 1, \ldots, M$. We will call the resulting FIS as ‘generalized-affine FIS’. Figure 2 shows an example of an FIS constructed as mentioned above.

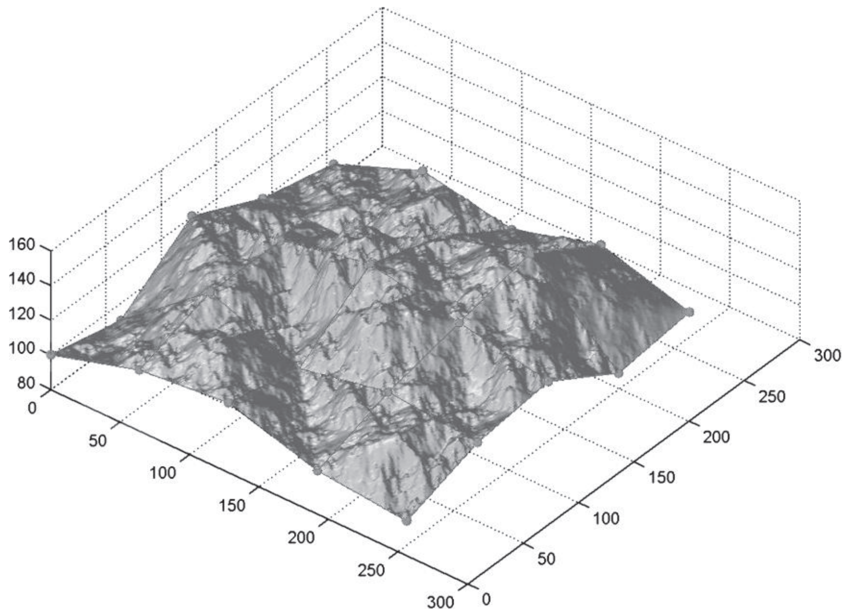


FIG. 1. An FIS that interpolates 5×5 interpolation points. The values of h at $\boldsymbol{x} \in \partial I_{i,j}$, $i = 1, 2, \ldots, 5$, $j = 1, 2, \ldots, 5$ (i.e. the 10 1D interpolants), are shown in red. In this case, we have selected 10 piecewise linear interpolants.

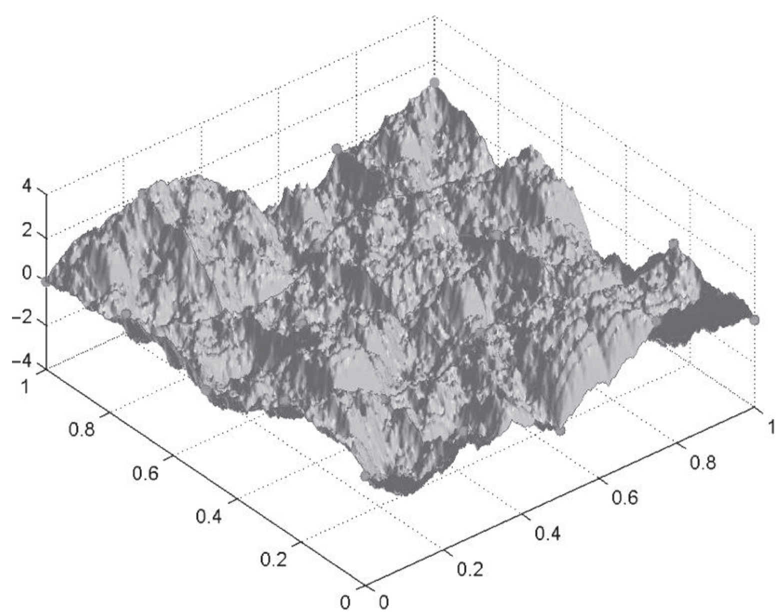


FIG. 2. An FIS that interpolates 5×5 interpolation points. The values of h at $\boldsymbol{x} \in \partial I_{i,j}$, $i = 1, 2, \ldots, 5$, $j = 1, 2, \ldots, 5$, are affine FIFs.

4. Computation of integrals and moments

To compute the inner product of two FIFs, we need to know the values of their moments. We note that for the 1D case, these values are already known (see Barnsley, 1986):

$$f_m = \int_0^1 x^m f(x)\mathrm{d}x = \frac{\sum_{k=0}^{m-1}\sum_{i=1}^{N}\binom{m}{k} a_i^{k+1} b_i^{m-k} s_i f_k + Q_m}{1-\sum_{i=1}^{N} a_i^{m+1} s_i}, \tag{23}$$

where $Q_m = \int_0^1 x^m Q(x)\mathrm{d}x$ and $Q(x) = p_i \circ T_i^{-1}(x)$, for $x \in I_i$. Hence, the inner product of two FIFs f and $\tilde{f}$ that interpolate the sets $\Delta = \{(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)\}$ and $\tilde{\Delta} = \{(x_0, \tilde{y}_0), (x_1, \tilde{y}_1), \ldots, (x_N, \tilde{y}_N)\}$ and are associated with the vertical scaling factors $\{s_1, \ldots, s_n\}$ and $\{\tilde{s}_1, \ldots, \tilde{s}_n\}$, respectively, is

$$\langle f, \tilde{f}\rangle = \int_0^1 f(x)\tilde{f}(x)\mathrm{d}x = \frac{\sum_{i=1}^{N} a_i s_i \int_0^1 f(x) p_i(x) + \sum_{i=1}^{N} a_i \tilde{s}_i \int_0^1 \tilde{f}(x) p_i(x) + \sum_{i=1}^{N} a_i \int_0^1 p_i(x)\tilde{p}_i(x)}{1-\sum_{i=1}^{N} a_i s_i \tilde{s}_i}, \tag{24}$$

where p_i and $\tilde{p}_i$ are the polynomials of the IFS maps (see Hardin *et al.*, 1992).

Using similar methods as in Barnsley (1986) and Hardin *et al.* (1992), one can compute the moments of an FIS defined on $[0, 1]^2$.

LEMMA 4.1. Let $f : [0, 1]^2 \to \mathbb{R}$ be a generalized-affine FIS that interpolates the points of $\Delta = \{(x_i, y_j, z_{i,j}) \in I \times \mathbb{R};\ i = 0, 1, \ldots, N,\ j = 0, 1, \ldots, M\}$, constructed as above. Then,

$$f_{n,m} = \int_{[0,1]^2} x^n y^m f(x, y)\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{\sum_{\substack{k=1,l=1\\(k,l)\neq(n,m)}}^{n,m} \sum_{i=1,j=1}^{N,M} \binom{n}{k}\binom{m}{l} a_i^{k+1} c_j^{l+1} b_i^{n-k} d_j^{m-l} s_{i,j} f_{k,l} + Q_{n,m}}{1-\sum_{i=1,j=1}^{N,M} a_i^{n+1} c_j^{m+1} s_{i,j}}, \tag{25}$$

where $Q_{n,m} = \int_{[0,1]^2} x^n y^m Q(x, y)\mathrm{d}x\,\mathrm{d}y$, $Q(x, y) = p_{i,j} \circ \boldsymbol{T}_{i,j}^{-1}(x, y)$, for $(x, y) \in I_{i,j}$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$.

Proof. Breaking the integral into parts and taking the functional relation (11), we have

$$f_{n,m} = \int_{[0,1]^2} x^n y^m f(x, y)\mathrm{d}x\,\mathrm{d}y = \sum_{i=1,j=1}^{N,M} \int_{I_{i,j}} x^n y^m f(x, y)\mathrm{d}x\,\mathrm{d}y$$

$$= \sum_{i=1,j=1}^{N,M} \int_{I_{i,j}} x^n y^m (s_{i,j} f(T_{1,i}^{-1}(x), T_{2,j}^{-1}(y)) + p_{i,j}(T_{1,i}^{-1}(x), T_{2,j}^{-1}(y))\mathrm{d}x\,\mathrm{d}y$$

$$= \sum_{i=1,j=1}^{N,M} a_i c_j \int_{[0,1]^2} (a_i x + b_i)^n (c_j y + d_j)^m \cdot (s_{i,j} f(x, y) + p_{i,j}(x, y))\mathrm{d}x\,\mathrm{d}y.$$

Applying Newton's binomial expansion formula and solving for $f_{n,m}$ gives the result. □

With this method one can compute several other integrals that are needed to compute the inner product of two FISs. The respective relations are given below without proof. Let $f(x,y)$ denote an FIS defined on $[0,1]^2$ as discussed above. Furthermore, let $u(y)$ and $\upsilon(x)$ be two FIFs defined on [0, 1], associated with the IFSs $\{\mathbb{R}, w^{(1)}_{1-M}\}$ and $\{\mathbb{R}, w^{(2)}_{1-N}\}$ and the interpolation points $\Delta^{(1)} = \{(y_j, z_j^{(1)}); j = 0,\ldots,M\}$ and $\Delta^{(2)} = \{(x_i, z_i^{(2)}); i = 0,\ldots,N\}$, respectively, where

$$w_j^{(1)}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} T_{2,j}(y) \\ \sigma_j z + q_j(y) \end{pmatrix} \quad \text{and} \quad w_i^{(2)}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} T_{1,i}(x) \\ \tilde{\sigma}_i z + \tilde{q}_i(x) \end{pmatrix},$$

for $i = 1,\ldots,N$, $j = 1,\ldots,M$. Similarly, let $\hat{u}(y)$ and $\hat{\upsilon}(x)$ be two FIFs defined on [0, 1], associated with the IFSs $\{\mathbb{R}, \hat{w}^{(1)}_{1-M}\}$ and $\{\mathbb{R}, \hat{w}^{(2)}_{1-N}\}$. Then,

- The integral $\int_{[0,1]^2} f(x,y)u(y)\mathrm{d}x\,\mathrm{d}y$ is computed as follows:

$$\int_{[0,1]^2} f(x,y)u(y)\mathrm{d}x\,\mathrm{d}y = \Bigg(\sum_{i=0}^{N}\sum_{j=0}^{M} a_{1,i}a_{2,j}s_{i,j}\int_{[0,1]^2} f(x,y)q_j(y)\mathrm{d}x\,\mathrm{d}y$$
$$+\sum_{i=0}^{N}\sum_{j=0}^{M} a_{1,i}a_{2,j}\sigma_j\int_{[0,1]^2} u(y)p_{i,j}(x,y)\mathrm{d}x\,\mathrm{d}y$$
$$+\sum_{i=0}^{N}\sum_{j=0}^{M} a_{1,i}a_{2,j}\int_{[0,1]^2} q_j(y)p_{i,j}(x,y)\mathrm{d}x\,\mathrm{d}y\Bigg)\Bigg/\Bigg(\sum_{i=0}^{N}\sum_{j=0}^{M} a_{1,i}a_{2,j}s_{i,j}\sigma_j\Bigg).$$

- The relation for the integral $\int_{[0,1]^2} f(x,y)\upsilon(x)\mathrm{d}x\,\mathrm{d}y$ is similar to the one above, with $\upsilon(x)$ in place of $u(y)$, $\tilde{q}_i$ in place of q_j and $\tilde{\sigma}_i$ in place of σ_j.
- The integral $\rho_{n,m} = \int_{[0,1]^2} x^n y^m u(y)\upsilon(x)\mathrm{d}x\,\mathrm{d}y$ is computed recursively as follows:

$$\rho_{n,m} = \Bigg(\sum_{i=0,j=0}^{N,M}\ \sum_{\substack{k=1,l=1\\(k,l)\neq(n,m)}}^{n,m}\binom{n}{k}\binom{m}{l} a_{1,i}^{k+1}a_{2,j}^{l+1}b_{1,i}^{n-k}b_{2,j}^{m-l}\sigma_i\tilde{\sigma}_j\rho_{k,l}$$
$$+\sum_{i=0,j=0}^{N,M}\sigma_i\int_{[0,1]^2}(a_{1,i}x+b_{1,i})^n(a_{2,j}y+b_{2,j})^m u(y)\tilde{q}_i(x)\mathrm{d}x\,\mathrm{d}y$$
$$+\sum_{i=0,j=0}^{N,M}\tilde{\sigma}_j\int_{[0,1]^2}(a_{1,i}x+b_{1,i})^n(a_{2,j}y+b_{2,j})^m \upsilon(x)q_j(y)\mathrm{d}x\,\mathrm{d}y$$
$$+\sum_{i=0,j=0}^{N,M}\tilde{\sigma}_j\int_{[0,1]^2}(a_{1,i}x+b_{1,i})^n(a_{2,j}y+b_{2,j})^m \tilde{q}_i(x)q_j(y)\mathrm{d}x\,\mathrm{d}y\Bigg)\Bigg/\Bigg(\sum_{i=0}^{N}\sum_{j=0}^{M} a_{1,i}^{\mu+1}a_{2,j}^{\nu+1}\sigma_j\tilde{\sigma}_i\Bigg),$$

where $\rho_{0,0}$ is given by

$$\rho_{0,0} = \left(\sum_{i=1,j=1}^{N,M} a_{1,i} a_{2,j} \sigma_j \int_{[0,1]^2} u(y) \tilde{q}_i(x) \mathrm{d}x\, \mathrm{d}y + \sum_{i=1,j=1}^{N,M} a_{1,i} a_{2,j} \tilde{\sigma}_i \int_{[0,1]^2} \upsilon(x) q_j(y) \mathrm{d}x\, \mathrm{d}y \right.$$

$$\left. + \sum_{i=1,j=1}^{N,M} a_{1,i} a_{2,j} \int_{[0,1]^2} \tilde{q}_i(x) q_j(y) \mathrm{d}x\, \mathrm{d}y \right) \Bigg/ \left(1 - \sum_{i=1,j=1}^{N,M} a_{1,i} a_{2,j} \tilde{\sigma}_i \sigma_j \right).$$

- The integral $\tau_{n,m} = \int_{[0,1]^2} x^n y^m \upsilon(x) \hat{\upsilon}(x) \mathrm{d}x\, \mathrm{d}y$ is computed recursively as follows:

$$\tau_{n,m} = \frac{1}{m+1} \left(\sum_{i=1}^{N} \sum_{k=0}^{n-1} \tilde{\sigma}_i \hat{\tilde{\sigma}}_i \binom{n}{k} a_{1,i}^{k+1} b_{1,i}^{n-k} \tau_{k,0} + \sum_{i=1}^{N} \tilde{\sigma}_i \int_{[0,1]^2} (a_{1,i} x + b_{1,i})^n \upsilon(x) \hat{\tilde{q}}_i(x) \mathrm{d}x\, \mathrm{d}y \right.$$

$$+ \sum_{i=1}^{N} \int_{[0,1]^2} \hat{\tilde{\sigma}}_i (a_{1,i} x + b_{1,i})^n \tilde{q}_i(x) \hat{\upsilon}(x) \mathrm{d}x\, \mathrm{d}y$$

$$\left. + \sum_{i=1}^{N} \int_{[0,1]^2} (a_{1,i} x + b_{1,i})^n \tilde{q}_i(x) \hat{\tilde{q}}_i(x) \mathrm{d}x\, \mathrm{d}y \right) \Bigg/ \left(1 - \sum_{i=1}^{N} \tilde{\sigma}_i \hat{\tilde{\sigma}}_i a_{1,i}^{n+1} \right).$$

- A similar relation holds for the integral $\tau'_{n,m} = \int_{[0,1]^2} x^n y^m u(y) \hat{u}(y) \mathrm{d}x\, \mathrm{d}y$.

The inner product of two FISs is given in the following proposition.

PROPOSITION 4.1 Consider two sets of interpolation points

$$\Delta = \{(x_i, y_j, z_{i,j}), i = 0, 1, \ldots, N, j = 0, 1, \ldots, M\},$$

$$\hat{\Delta} = \{(x_i, y_j, \hat{z}_{i,j}), i = 0, 1, \ldots, N, j = 0, 1, \ldots, M\}$$

such that $0 = x_0 < x_1 < \cdots < x_N = 1$ and $0 = y_0 < y_1 < \cdots < y_M = 1$. Let u_i, $\hat{u}_i$ be the 1D interpolants associated with the sets Δ_i and $\hat{\Delta}_i = \{(x_i, y_j, \hat{z}_{i,j}), j = 0, 1, \ldots, M\}$, respectively. Similarly, let υ_j, $\hat{\upsilon}_j$ be the interpolants associated with the sets $\tilde{\Delta}_j$ and $\hat{\tilde{\Delta}}_j = \{(x_i, y_j, z_{i,j}), i = 0, 1, \ldots, N\}$. Consider the FISs f, $\hat{f}$ that interpolate Δ and $\hat{\Delta}$ and are associated with the borders u_i, υ_j and $\hat{u}_i, \hat{\upsilon}_j$ for $i = 0, \ldots, N$, $j = 0, \ldots, M$, with vertical scaling factors $s_{i,j}$ and $\hat{s}_{i,j}$, respectively, for $i = 1, \ldots, N$, $j = 1, \ldots, M$. Then the inner product of f and $\hat{f}$ is given by

$$\int_{[0,1]^2} f(x,y) \hat{f}(x,y) \mathrm{d}y\, \mathrm{d}x = \frac{1}{1 - \sum_{i=1}^{N} \sum_{j=1}^{M} s_{i,j} \hat{s}_{i,j} a_i c_j} \cdot$$

$$\left(\sum_{i=1}^{N} \sum_{j=1}^{M} \hat{s}_{i,j} \left(\int_{[0,1]^2} \hat{f}(x,y) p_{i,j}(x,y) \mathrm{d}y\, \mathrm{d}x \right) a_i c_j \right.$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{M} s_{i,j}\left(\int_{[0,1]^2} f(x,y)\hat{p}_{i,j}(x,y)\mathrm{d}y\,\mathrm{d}x\right)a_i c_j$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{M}\left(\int_{[0,1]^2} p_{i,j}(x,y)\hat{p}_{i,j}(x,y)\mathrm{d}y\,\mathrm{d}x\right)a_i c_j\Bigg),$$

where $p_{i,j}$ and $\hat{p}_{i,j}$ are given by (22) in each case.

Putting together Proposition 4.1 and relation (22), it is evident that in order to compute the inner product $\int_{[0,1]^2} f(x,y)\hat{f}(x,y)\mathrm{d}x\,\mathrm{d}y$ we need to compute several integrals of the form

$$\int_{[0,1]^2} x^n y^m u_i(y)\upsilon_j(x)\mathrm{d}x\,\mathrm{d}y,\quad \int_{[0,1]^2} x^n y^m u_i(y)u_{i'}(y)\mathrm{d}x\,\mathrm{d}y,\quad \int_{[0,1]^2} x^n u_i(y) f(x,y)\mathrm{d}x\,\mathrm{d}y,$$

$$\int_{[0,1]^2} x^n y^m u_i(T_{2,l}(y))\upsilon_j(x)\mathrm{d}x\,\mathrm{d}y,\quad \int_{[0,1]^2} x^n y^m u_i(T_{2,l}(y))u_{i'}(y)\mathrm{d}x\,\mathrm{d}y,$$

$$\int_{[0,1]^2} x^n u_i(T_{2,l}(y)) f(x,y)\mathrm{d}x\,\mathrm{d}y$$

(and likewise for υ_j and $\upsilon_j \circ T_{1,k}$) and the moments of u_i, υ_j, $u_i \circ T_{2,l}$, $\upsilon_j \circ T_{1,k}$, for all possible combinations of i, i', j, j', k, l and $n, m = 0, 1$. The first group of integrals can be evaluated using the relations presented above in this section. For the second group, we need to observe that $u_i(T_{2,l}(y))$ and $\upsilon_j(T_{1,k}(x))$ are also affine FIFs, for all i, j (see Proposition 5.1 in Section 5), and then use the same relations. After considerable algebra (which can be done by Mathematica or Maple), we take the inner product as a linear combination of the products $z_{i,j} \cdot \hat{z}_{k,l}$, for $i, k = 0, 1, \ldots, N$, $j, l = 0, 1, \ldots, M$. The coefficients of $z_{i,j} \cdot \hat{z}_{k,l}$ will be polynomials of the vertical scaling factors $s_{i,j}$, $\hat{s}_{k,l}$, $i, k = 0, 1, \ldots, N$, $j, l = 0, 1, \ldots, M$.

5. Dilation properties of FIFs

We have already mentioned that affine FIFs satisfy certain dilation properties. The aim of this section is to prove that similar relations are true for the generalized-affine FISs. In the following, we will limit our interest to FISs that are constructed taking into account that the 1D interpolants are affine FIFs, as mentioned in the last lines of Section 3. In addition, we will assume that the vertical scaling factors used for the construction of the affine FIFs and the construction of the FIS are equal to s (i.e. $s_{i,j} = s$, $i = 1, \ldots, N$, $j = 1, \ldots, M$; $\sigma_{i,j} = s$, $i = 0, \ldots, N$, $j = 1, \ldots, M$; $\tilde{\sigma}_{i,j} = s$, $i = 1, \ldots, N$, $j = 0, \ldots, M$). In the rest of the paper, when we are refereing to a generalized-affine FIS, we will mean an FIS constructed in this manner (unless it is explicitly stated otherwise).

5.1 *Dilation properties of affine FIFs*

For the case of the affine FIF, it has been noticed that certain dilation properties hold. In particular, if we restrict an FIF that interpolates N points (see Section 2.2) on the interval $[x_{i-1}, x_i]$, then we get another FIF. This property is described in the following proposition. Its proof makes use of the self-affiniteness of the graph of f (see Hardin *et al.*, 1992).

PROPOSITION 5.1 Let f be an affine FIF associated with the set of interpolation points $\Delta = \{(x_i, y_i), i = 0, 1, \ldots, N\}$ and the vertical scaling factor s (see Section 2.2). Let $w_1, \ldots, w_N$ be the affine mappings that form the respective IFS, then for any $k = 1, \ldots, N$,

(i) The restriction of f on $[x_{k-1}, x_k]$ is also an affine FIF that is associated with the points $\{(T_k(x_i), F_k(x_i, y_i)), i = 0, 1, \ldots, N\}$ and the vertical scaling factor s. The associated IFS contains the affine mappings $w_k \circ w_i \circ w_k^{-1}, i = 1, \ldots, N$.

(ii) The function $f(T_k(x))$ is also an affine FIF, associated with the points $\{(x_i, w_k(x_i)), i=0, \ldots, N\}$ and the vertical scaling function s. The associated IFS contains the mappings

$$w_i'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} T_i(x) \\ s_i y - sp_k(x) + sp_i(x) + p_k(T_i(x)) \end{pmatrix},$$

for $i = 1, \ldots, N$.

Proof.

1. Let G be the graph of f, then $w_k(G)$ is the graph of the restriction of f on $[x_{k-1}, x_k]$. The result now follows from the fact that

$$G = \bigcup_{i=1}^{N} w_i(G) \Rightarrow w_k(G) = \bigcup_{i=1}^{N} w_k \circ w_i \circ w_k^{-1}(w_k(G)).$$

2. The graph of $f(T_k(x))$ is $\check{w}_k \circ w_k(G)$, where $\check{w}_k$ is given by

$$\check{w}_k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} T_k^{-1}(x) \\ y \end{pmatrix}.$$

Similarly to the first part, we easily obtain

$$\check{w}_k \circ w_k(G) = \bigcup_{i=1}^{N} \check{w}_k \circ w_k \circ w_i \circ \circ w_k^{-1} \check{w}_k(\check{w}_k \circ w_k(G)).$$

The result follows after some algebra. □

The following result is also found in Hardin *et al.* (1992) and will be used later.

PROPOSITION 5.2 Let $\mathcal{P} = \{x_i, i = 0, 1, \ldots, N\}$, $0 = x_0 < x_1 < \cdots < x_N = 1$, be a partition of $[0, 1]$ and $|s| < 1$. The space $\mathbb{F}_{\mathcal{P},s}$ of all affine FIFs that interpolate points of the form $\Delta = \{(x_i, y_i), i = 0, 1, \ldots, N\}$ and are associated with vertical scaling factor s is a linear space with dimension $N + 1$.

5.2 *Dilation properties of FISs*

Similar results hold for the generalized-affine FISs.

PROPOSITION 5.3 Let the set of points $\mathcal{P} = \{(x_i, y_j), i = 0, \ldots, N, j = 0, 1, \ldots, M\}$, such that $0 = x_0 < x_1 < \cdots < x_N = 1$ and $0 = y_0 < y_1 < \cdots < y_M = 1$, that define a partition of $[0, 1]^2$ and $|s| < 1$. The set $\mathbb{F}_{\mathcal{P},s}$ of the generalized-affine FIS that interpolate a set of points of the form $\Delta = \{(x_i, y_j, z_{i,j}), i = 0, 1, \ldots, N, j = 0, 1, \ldots, M\}$ and are associated with the vertical scaling factor s is a linear space with dimension $(N + 1) \cdot (M + 1)$.

Proof. Let $f_1, f_2 \in \mathbb{F}_{\mathcal{P},s}$, two generalized-affine FISs with graphs G_1 and G_2 that correspond to interpolation points $\varDelta_1$ and $\varDelta_2$, respectively, and $\lambda_1, \lambda_2 \in \mathbb{R}$. Let $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$ be the mappings of the associated IFSs (see (6)) and $\hat{G}$ be the graph of the function $\hat{f} = \lambda_1 \cdot f_1 + \lambda_2 \cdot f_2$. If $u^{(1)}_i, \upsilon^{(1)}_j$ and $u^{(2)}_i, \upsilon^{(2)}_j$ are the corresponding affine FIFs used in the construction of f_1 and f_2, then Proposition 5.2 ensures that the 1D interpolants $\hat{u}_i = \lambda_1 \cdot u^{(1)}_i + \lambda_2 \cdot u^{(2)}_i$ and $\hat{\upsilon}_j = \lambda_1 \cdot \upsilon^{(1)}_j + \lambda_2 \cdot \upsilon^{(2)}_j$ are affine FIFs (for $i = 0, \ldots, N$, $j = 0, \ldots, M$). It is easy to prove that the mappings $\hat{W}_{i,j}$ defined by

$$\hat{W}_{i,j}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} T_{1,i}(x) \\ T_{2,j}(y) \\ s \cdot z + \hat{p}_{i,j}(x, y) \end{pmatrix},$$

with $\hat{p}_{i,j}(x, y) = \lambda_1 \cdot p^{(1)}_{i,j}(x, y) + \lambda_2 \cdot p^{(2)}_{i,j}(x, y)$, where $p^{(1)}_{i,j}$ and $p^{(2)}_{i,j}$ are the corresponding functions of $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$, for $i = 1, \ldots, N$, $j = 1, \ldots, M$, satisfy the relation $\hat{G} = \bigcup_{i=1}^{N} \bigcup_{j=1}^{M} \hat{W}_{i,j}(\hat{G})$. Furthermore, we can easily verify that $\hat{p}_{i,j}$ satisfies (22), with $\hat{u}_i$, $\hat{\upsilon}_j$ in place of u_i, υ_j and $\hat{z}_{i,j} = \lambda_1 \cdot z^{(1)}_{i,j} + \lambda_2 \cdot z^{(2)}_{i,j}$ in place of $z_{i,j}$, for $i = 1, \ldots, N$, $j = 1, \ldots, M$. A straightforward choice for the base of this linear space are the $(N + 1) \cdot (M + 1)$ functions obtained by putting a '1' on each of the $(N + 1) \cdot (M + 1)$ interpolation points and filling the rest with zeros. □

PROPOSITION 5.4 Let $\varDelta = \{(x_i, y_j, z_{i,j}); i = 0, \ldots, N, j = 0, \ldots, M\}$ be a set of interpolation points and $|s| < 1$. Let f be the generalized-affine FIS that interpolates the points of $\varDelta$. Then the restriction of f on $I_{k,l} = [x_{k-1}, x_k] \times [y_{l-1}, y_l]$ and $g = f|_{I_{k,l}}$ (for fixed k, l) is also a generalized-affine FIS interpolating the set of points

$$\varDelta_{k,l} = \{(T_{1,k}(x_i), T_{2,l}(y_j), F_{k,l}(x_i, y_j, z_{i,j})); i = 0, \ldots, N, j = 0, \ldots, M\}.$$

Proof. We split the proof into two parts. In the first, we will prove that the affine FIFs u_i, υ_j, $i = 0, \ldots, N$, $j = 0, \ldots, M$, are mapped (through $W_{k,l}$) to affine FIFs. Subsequently, we will deduce that the graph of g satisfies a relation such as (12), where the mappings are similar to (22).

For the first part, let u_i be one of the affine FIFs that interpolate the points of $\varDelta_i = \{(x_i, y_j, z_{i,j}); j = 0, \ldots, M\}$ and let $\tilde{u}_i$ be a function defined on $[y_{k-1}, y_k]$ such that $\tilde{u}_i(y) = F_{k,l}(x_i, T^{-1}_{2,l}(y), u_i(T^{-1}_{2,j}(y)))$ for all $i = 0, \ldots, N$. Substituting x with x_i in (22), we can easily see that $\tilde{u}_i$ is expressed as a linear combination of affine FIFs (plus a constant function which is an affine FIF).

For the final part, we observe that

$$W_{k,l}(G) = \bigcup_{i=1}^{N} \bigcup_{j=1}^{M} W_{k,l} \circ W_{i,j} \circ W^{-1}_{k,l}(W_{k,l}(G)), \tag{26}$$

where G is the graph of f and $W_{k,l}(G)$ the graph of g. Considering that

$$W^{-1}_{k,l}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} T^{-1}_{1,k}(x) \\ T^{-1}_{2,l}(y) \\ \frac{1}{s}(z - p_{k,l}(T^{-1}_{1,k}(x), T^{-1}_{2,l}(y))) \end{pmatrix},$$

it takes a few lines of algebra to see that the mappings $\hat{W}_{i,j} = W_{k,l} \circ W_{i,j} \circ W_{k,l}^{-1}$ have the form

$$\hat{W}_{i,j}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}T_{1,k} \circ T_{1,i} \circ T_{1,k}^{-1}(x)\\ T_{2,l} \circ T_{2,j} \circ T_{2,l}^{-1}(y)\\ s \cdot z + \hat{p}_{i,j}(x, y)\end{pmatrix},$$

where

$$\hat{p}_{i,j}(x, y) = -s \cdot p_{k,l}(T_{1,k}^{-1}(x), T_{2,l}^{-1}(y)) + s \cdot p_{i,j}(T_{1,k}^{-1}(x), T_{2,l}^{-1}(y)) + p_{k,l}(T_{1,i} \circ T_{1,k}^{-1}(x), T_{2,i} \circ T_{2,l}^{-1}(y)),$$

for $i = 1, \dots, N,\ j = 1, \dots, M$. It is readily proved that $\hat{W}_{i,j}$ has the form (6) and $\hat{p}_{i,j}$ has the form (17) for all i, j. Therefore, taking into account the first part of the proof and the fact that all $\hat{W}_{i,j}$ satisfy (26), we deduce that the function g is a generalized-affine FIS. □

6. Multiresolution analysis obtained from FISs

In the following, we give a definition of multiresolution analysis based on N-adic dilates (where $N \geqslant 2$), instead of dyadic ones which are more widely used. A 'multiresolution analysis of multiplicity r' of $L_2(\mathbb{R}^2)$ is a nested sequence of closed linear subspaces (V_n) in $L_2(\mathbb{R}^2)$ satisfying the following:

(A) $f \in V_k$, if and only if $f(N^{-k}\cdot) \in V_0$.

(B) Nestedness. $V_0 \subset V_1$.

(C) Density. $\overline{\bigcup_{k\in\mathbb{Z}} V_k} = L_2(\mathbb{R}^2)$.

(D) Separation. $\bigcap_{k\in\mathbb{Z}} V_k = \{0\}$.

(E) Stable shifts. There are r functions $\phi^1, \phi^2, \dots, \phi^r$ such that the collection of integer translates $\{\phi^\alpha_{0,i,j} = \phi^\alpha(\cdot - i, \cdot - j)/\alpha = 1, \dots, r, i, j \in \mathbb{Z}\}$ forms a Riesz basis of V_0.

An immediate consequence of the above relations is that the set

$$\{\phi^\alpha_{k,i,j} = \phi^\alpha(N^k \cdot - i, N^k \cdot - j); \alpha = 1, \dots, r, i, j \in \mathbb{Z}\}$$

is a Riesz basis of V_k. The functions $\phi^1, \dots, \phi^r$ are called scaling functions and are said to generate the multiresolution analysis. The vector function $\Phi = (\phi^1, \dots, \phi^r)^\top$ is called scaling vector. If there is a set of compactly supported scaling functions whose integer translates form an orthogonal basis of V_0, then we call (V_k) 'orthogonal' multiresolution analysis. We note that since the number of the scaling functions is finite, V_0 is a 'finitely generated shift-invariant' (FSI) space. Another immediate consequence of the above relations is that Φ satisfies a 'matrix-vector refinement equation' of the form

$$\Phi(x, y) = \sum_{i,j\in\mathbb{Z}} C_{i,j}\Phi(Nx - i, Ny - j), \tag{27}$$

for some sequence of $r \times r$ matrices $C_{i,j}$, called 'scaling coefficients'.

There are several results regarding the conditions that the functions $\phi^1, \dots, \phi^r$ need to satisfy, so that they generate a multiresolution analysis. Conditions for the density property (C) were given by de Boor, DeVore and Ron and can be found in de Boor *et al.* (1993) for the case where $r = 1$, but can be easily extended (see also Jia & Shen, 1994). Their result can be stated as follows (for $r = 1$): If $\mathcal{F}(\phi)$ (i.e. the Fourier transform of ϕ) is non-zero almost everywhere in some neighbourhood of the origin, then the density property holds. Note that in the case where ϕ has compact support, this condition is true. For the separation property, we have the general result given by Jia & Shen (1994): Any FSI subspace of $L_2(\mathbb{R}^d)$ satisfies (D).

For the purpose of our construction, we fix N and s and define V_0 to be the space consisting of functions $f \in L_2(\mathbb{R}^2)$, whose restriction to $[\alpha, \alpha+1]\times[\beta, \beta+1]$ is a generalized-affine FIS interpolating sets of points of the form

$$\Delta = \{(\alpha + i/N, \beta + j/N, z_{i,j}); i, j = 0, \ldots, N\},$$

for all $\alpha, \beta \in \mathbb{N}$. The corresponding $(N+1)^2$ scaling functions and their translates must form a basis of V_0. One such base can be obtained by selecting ϕ^κ, where $\kappa = l \cdot (N+1) + k + 1$, as the FIS associated with the set of points

$$\Delta_{k,l} = \{(i/N, j/N, z_{i,j}) \colon z_{k,l} = \delta_{i,k}\delta_{j,l}\},$$

for all $k, l = 0, \ldots, N$. It is easy to verify that the corresponding scaling vector Φ will satisfy a refinement equation such as (27).

PROPOSITION 6.1 Consider the generalized-affine FISs $\phi^1, \ldots, \phi^r$, defined as above (where $r = (N+1)^2$). Let $\{X, W^\kappa_{i,j}, i, j = 1, \ldots, N\}$ be the IFS related to ϕ^κ, for $\kappa = 1, \ldots, r$ (where $X = [0, 1]^2 \times \mathbb{R}$ and $W^\kappa_{i,j} = (T_{1,i}, T_{2,j}, F^k_{i,j})^\top$, see also Section 3). Then the vector $\Phi = (\phi^1, \ldots, \phi^r)^\top$ satisfies the refinement equation

$$\Phi(x, y) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} C_{i,j} \cdot \Phi(Nx - i, Ny - j), \tag{28}$$

where $C_{i,j}$ are $r \times r$ matrices whose elements are given by

$$C_{i,j}(\kappa, \lambda) = F^\kappa_{i,j}(k/N, l/N, z_{k,l}),$$

with k, l the unique integers satisfying $\lambda = l \cdot (N+1) + k + 1$ (i.e. $l = (\lambda - 1)\,\mathrm{div}(N+1)$, $k = (\lambda - 1) \bmod (N+1)$), for all $i, j = 0, \ldots, N-1$, $\kappa, \lambda = 1, \ldots, r$).

Proof. Let κ be fixed. By Proposition 5.4, we know that the restriction of ϕ^κ in each of the sets $I_{i,j} = [i/N, (i+1)/N]\times[j/N, (j+1)/N]$, $i, j = 1, \ldots, N$, is also a generalized-affine FIS. Thus, $\phi^\kappa|_{I_{i,j}}$ can be expressed as a linear combination of $\phi^1(N\cdot -i, N\cdot -j), \ldots, \phi^r(N\cdot -i, N\cdot -j)$, with its coefficient values on the vertices of the corresponding grid

$$\phi^\kappa(x, y) = \sum_{\lambda=1}^{r} F^\kappa_{i,j}(k/N, l/N, z_{k,l}) \cdot \phi^\lambda(Nx - i, Ny - j),$$

for all $(x, y) \in I_{i,j}$, where $l = (\lambda - 1)\,\mathrm{div}(N+1)$ and $k = (\lambda - 1) \bmod (N+1)$. The result follows immediately. □

Using the Gram–Schmidt orthogonalization process, we may obtain an orthonormal base of V_0, namely $\hat{\phi}^1, \ldots, \hat{\phi}^r$. We define V_k as the space produced by $\hat{\phi}^1(N^k\cdot, N^k\cdot), \ldots, \hat{\phi}^r(N^k\cdot, N^k\cdot)$ and their translates, i.e.

$$V_k = \overline{\mathrm{span}\{\hat{\phi}^\kappa(N^k \cdot -i, N^k \cdot -j); i, j \in \mathbb{Z}, \kappa = 1, \ldots, r\}}.$$

The following is true.

PROPOSITION 6.2 The spaces V_k, $k \in \mathbb{Z}$, generate a multiresolution analysis of $L_2(\mathbb{R}^2)$.

Proof. Conditions (A) and (B) clearly hold due to the construction (see (28) and Proposition 5.4). Since V_0 is an FSI space, conditions (C) and (D) also hold as mentioned above. The last condition follows from the orthogonality of $\hat{\phi}^1, \ldots, \hat{\phi}^r$. □

As is usually the case, we define W_k as the orthogonal complement of V_k into V_{k+1}, i.e. $V_k \oplus W_k = V_{k+1}$. In this case, all the W_k's are scaled versions of W_0 (i.e. $f \in W_k \Leftrightarrow f(N^{-k}) \in W_0$), $W_k \perp W_{k'}$, for $k \neq k'$, and $L_2(\mathbb{R}^2) = \overline{\oplus_{k \in \mathbb{Z}} W_k}$. In addition, there exist functions $\psi^1, \ldots, \psi^{r'}$ ($r' = (N^2 - 1)(N + 1)^2$ in our case) orthogonal to ϕ's and to each other so that their integer translates form a Riesz basis of W_0. The functions $\psi^1, \ldots, \psi^{r'}$ are called multi-wavelets. The wavelet vector $\Psi = (\psi^1, \ldots, \psi^{r'})$ will satisfy a relation of the form

$$\Psi(x, y) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} D_{i,j} \cdot \Phi(Nx - i, Ny - j), \tag{29}$$

where $D_{i,j}$ are $r \times r$ matrices, for $i, j = 0, \ldots, N - 1$. The wavelet coefficients $D_{i,j}$ can be computed by solving the linear system

$$\langle \psi^k(x, y), \phi^l(x, y) \rangle = 0,$$

for $k = 1, \ldots, r'$, $l = 1, \ldots, r$. Let $C_{i,j}(l, \lambda)$ denote the element of the $C_{i,j}$ matrix positioned at lth line, λth column and $D_{i,j}(k, \kappa)$ denote the element of $D_{i,j}$ positioned at kth line, κth column. Then the above linear system can be reformulated as

$$\left\langle \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{\kappa=1}^{r'} D_{i,j}(k, \kappa) \cdot \phi^\kappa(Nx - i, Ny - j), \sum_{i'=0}^{N-1} \sum_{j'=0}^{N-1} \sum_{\lambda=1}^{r} C_{i',j'}(l, \lambda) \cdot \phi^\lambda(Nx - i', Ny - j') \right\rangle = 0,$$

or equivalently

$$\sum_{\kappa=1}^{r'} \sum_{\lambda=1}^{r} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} C_{i,j}(l, \lambda) \cdot \langle \phi^\kappa(Nx - i, Ny - j), \phi^\lambda(Nx - i, Ny - j) \rangle \cdot D_{i,j}(k, \kappa) = 0, \tag{30}$$

for all $k = 1, \ldots, r'$, $l = 1, \ldots, r$. We can then apply the Gram–Schmidt orthogonalization procedure to obtain an orthonormal basis. Another more elegant approach is to extend the polyphase matrix in such a way that the extended matrix is paraunitary (see, e.g. Strang & Strela, 1995; Lawton *et al.*, 1996; Vaidyanathan, 1993; Keinert, 2004). Either way, the resulting multi-wavelets are not unique. At this point we should note that the multi-wavelets are not continuous functions. They will have possible discontinuities at the points $[0, 1] \times \{i/N\}$ and $\{i/N\} \times [0, 1]$, for all $i = 0, \ldots, N$.

REMARK 6.1. It is easy to prove that the scaling vector has accuracy 2 (i.e. any polynomial up to order 1 belongs to V_0). This means that the wavelet functions will have two vanishing moments.

For the implementation of the discrete multi-wavelet transform, we must provide a suitable prefiltering (preprocessing) technique. Prefiltering is the process of converting the equally spaced samples of a given signal $s(x, y) \in V_0$ to the vector coefficients $S_{i,j}$ appearing in the multi-scaling expansion of the signal, i.e.

$$s(x, y) = \sum_{i,j} S_{i,j}^\top \cdot \Phi(x - i, y - j).$$

For scalar wavelets, the preprocessing and postprocessing steps are often omitted and the expansion coefficients are equated with the point samples. This is often called as a ‘wavelet crime’ (see Strang & Nguyen, 1996). For the multi-wavelets, however, preprocessing and postprocessing steps are necessary. We will use a short of interpolating prefilter (Hardin *et al.*, 1992; Xia *et al.*, 1998). We assume that the signal has been sampled at the points $(x_{i,k}, y_{j,l})$, $x_{i,k} = i + (k/N)$, $y_{j,l} = j + (l/N)$, for $k, l = 0, \ldots, N-1$. Then, the κth element of the vector $S_{i,j}$ is given by

$$S_{i,j}(\kappa) = s(x_{i,k}, y_{j,l}),$$

where k, l such that $\kappa = l(N+1) + (k+1)$.

6.1 *An example*

Consider the case where $N=2$, i.e. we have nine scaling functions. The inner product of two generalized-affine FIFs f and $\hat{f}$ each one of them interpolating the data sets

$$\Delta = \left\{\left(\frac{i}{2}, \frac{j}{2}, z_{i,j}\right); i, j = 0, 1, 2\right\} \text{ and } \hat{\Delta} = \left\{\left(\frac{i}{2}, \frac{j}{2}, \hat{z}_{i,j}\right); i, j = 0, 1, 2\right\},$$

respectively, will be given as a linear combination of the products $z_{i,j} \cdot \hat{z}_{i,j}$, $i, j = 0, \ldots, 2$, where the coefficients are rational polynomials of s, with the common denominator

$$1152(-1+s)^4(1+s)^3.$$

Although it is possible to give the exact formula using the techniques in Section 4, we chose to omit it since it is very large and does not provide any additional information (the computation was implemented in Mathematica). The scaling coefficients $C_{i,j}$, $i, j = 0, 1$, are given below:

$$C_{0,0} = \begin{pmatrix} 1 & \frac{1-s}{2} & 0 & \frac{1-s}{2} & \frac{1-s}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}+s & 1 & 0 & \frac{1-s}{4} & \frac{1-s}{2} & 0 & 0 & 0 \\ 0 & -\frac{s}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}+s & \frac{1-s}{4} & 0 & 1 & \frac{1-s}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1+8s}{4} & \frac{1}{2}+s & 0 & \frac{1}{2}+s & 1 \\ 0 & 0 & 0 & -\frac{s}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3s}{4} & -\frac{s}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{s}{4} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_{1,0} = \begin{pmatrix} 0 & -\frac{s}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1+2s}{2} & 0 & \frac{1-s}{2} & \frac{1-s}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1-s}{2} & 1 & 0 & \frac{1-s}{4} & \frac{1-s}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3s}{4} & 0 & 0 & -\frac{s}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2}+s & \frac{1+8s}{4} & 0 & 1 & \frac{1+2s}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-s}{4} & \frac{1}{2}+s & 0 & \frac{1-s}{2} & 1 \\ 0 & 0 & 0 & -\frac{s}{2} & -\frac{3s}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{s}{2} & -\frac{3s}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{s}{2} & 0 & 0 & 0 \end{pmatrix},$$

$$C_{0,1} = \begin{pmatrix} 0 & 0 & 0 & -\frac{s}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3s}{4} & -\frac{s}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{s}{4} & 0 & 0 & 0 & 0 \\ 1 & \frac{1-s}{2} & 0 & \frac{1-2s}{2} & \frac{1-s}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}+s & 1 & 0 & \frac{1+8s}{4} & \frac{1+2s}{2} & 0 & 0 & 0 \\ 0 & -\frac{s}{2} & 0 & 0 & -\frac{3s}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-s}{2} & \frac{1-s}{4} & 0 & 1 & \frac{1-s}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-s}{4} & \frac{1-s}{2} & 0 & \frac{1}{2}+s & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{s}{2} & 0 \end{pmatrix},$$

$$C_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{s}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{s}{2} & -\frac{3s}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{s}{2} & 0 & 0 & 0 \\ 0 & -\frac{s}{2} & 0 & 0 & -\frac{3s}{4} & 0 & 0 & 0 & 0 \\ 1 & \frac{1+2s}{2} & 0 & \frac{1+2s}{2} & \frac{1+8s}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1-s}{2} & 1 & 0 & \frac{1-s}{4} & \frac{1+2s}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{s}{2} & 0 \\ 0 & 0 & 0 & \frac{1-s}{2} & \frac{1-s}{4} & 0 & 1 & \frac{1+2s}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-s}{4} & \frac{1-s}{2} & 0 & \frac{1-s}{2} & 1 \end{pmatrix}.$$

We can find the corresponding wavelets using Mathematica or Maple to extend the polyphase matrix or solve the linear system and orthonormalize as mentioned earlier. It is possible to give an exact formula of one set of corresponding wavelets in terms of s, but since it is very long we rather give an approximation for the case $s = \frac{1}{2}$ (see Tables 1 and 2).

Figure 3 shows the scaling functions, while Figs 4 and 5 show the corresponding orthonormal multiwavelets.

7. Generalization of the construction to $L_2(\mathbb{R}^d)$

In this section, we generalize the construction using FIFs defined on $[0, 1]^d$, $d \geqslant 3$. The key ideas are the same as before. However, we need some results regarding a Hölder property of a general class of FIFs in order to make the construction valid.

7.1 *FIFs on rectangular lattices of* $[0, 1]^d$

Consider a set of interpolation points of the form

$$\Delta = \left\{ \left(x_{1,i_1}, x_{2,i_2}, \ldots, x_{d,i_d}, z_{i_1,i_2,\ldots,i_d}\right) \in [0,1]^d \times \mathbb{R};\, i_l = 0, \ldots, N_l, l = 1, 2, \ldots, d \right\}$$

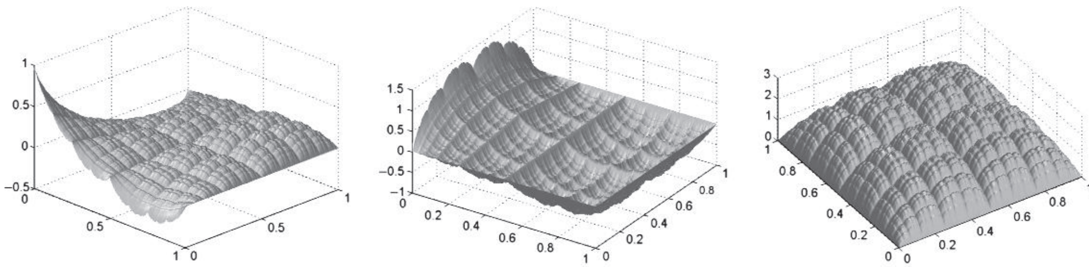
such that $0 = x_{l,0} < x_{l,1} < \cdots < x_{l,N_l} = 1$, $N_l \in \mathbb{N}$, for $l = 1, \ldots, d$, which contains in total $(N_1+1) \cdot (N_2+1) \cdots (N_d+1) = \prod_{l=1}^{d} (N_l+1)$ points. To simplify the notation, we set $\boldsymbol{i} = (i_1, i_2, \ldots, i_n) \in \mathbb{A}_0$,

TABLE 1 *The wavelet coefficients of example 6.1 for* $s = 1/2$, *part A.* $d_{i,j,k,\kappa} := D_{i,j}(k,\kappa)$

k	$d_{0,0,k,1}$	$d_{0,0,k,2}$	$d_{0,0,k,3}$	$d_{0,0,k,4}$	$d_{0,0,k,5}$	$d_{0,0,k,6}$	$d_{0,0,k,7}$	$d_{0,0,k,8}$	$d_{0,0,k,9}$	$d_{0,1,k,1}$	$d_{0,1,k,2}$	$d_{0,1,k,3}$	$d_{0,1,k,4}$	$d_{0,1,k,5}$	$d_{0,1,k,6}$	$d_{0,1,k,7}$	$d_{0,1,k,8}$	$d_{0,1,k,9}$
1	4.83	0	1.16	0	−0.607	0	0	0	0	2.17	0	0	0	0.184	1.16	0	0	0
2	−4.07	1.52	1.48	1.52	0.658	0	0	0	0	−7.69	0	0	0	0.824	1.48	1.52	0	0
3	1.20	−1.35	−0.903	0.845	0.522	2.20	0	0	0	3.96	0	0	0	−0.425	−0.903	0.845	2.20	0
4	8.50	−0.756	0.132	−0.623	−1.52	0.133	8.59	0	0	−10.4	0	0	0	0.170	0.132	−0.623	0.133	8.59
5	1.36	−0.485	−0.0544	1.69	0.805	0.541	−3.34	1.63	0	−8.50	0	0	0	0.698	−0.0544	1.69	0.541	−3.34
6	6.05	0.715	1.49	−1.51	−0.535	−0.280	−2.32	1.61	3.56	−0.380	0	0	0	0.705	1.49	−1.51	−0.280	−2.32
7	5.62	0.775	−1.41	0.917	−0.450	−0.00154	−0.686	−0.0587	−0.202	2.35	2.87	0	0	0.176	−1.41	0.917	−0.00154	−0.686
8	−0.838	−1.26	−0.318	−1.10	0.0833	0.991	−3.58	−1.71	−0.882	−2.48	1.95	3.08	0	0.855	−0.318	−1.10	0.991	−3.58
9	5.53	1.14	2.05	−0.849	−0.328	−0.500	−1.89	0.240	−2.42	2.73	0.0495	0.326	4.14	0.761	2.05	−0.849	−0.500	−1.89
10	1.61	0.942	2.49	2.26	1.16	3.99	−0.0307	1.08	0.0230	5.10	−0.920	0.0236	−2.06	−2.55	−3.30	8.04	−1.80	−0.0307
11	4.86	−0.789	6.14	0.976	−0.721	1.86	0.247	−1.28	−0.846	1.89	−0.352	−0.230	1.47	0.514	3.84	3.28	−0.441	0.247
12	0.951	2.44	2.80	−0.927	0.0326	0.383	0.132	0.529	0.213	0.774	−0.948	0.846	−0.488	−0.964	−0.490	2.36	−2.91	0.132
13	−2.26	−0.184	2.85	0.973	0.747	1.72	−5.25	−0.374	0.170	2.38	0.457	−0.974	1.29	0.668	1.40	2.42	0.266	−5.25
14	−1.52	−1.00	0.604	−0.477	−0.881	0.482	1.40	−2.41	3.05	−0.785	0.138	0.330	0.297	0.0319	0.318	−0.191	0.196	1.40
15	4.36	−0.145	−0.842	0.549	0.513	−0.812	0.367	−0.0537	−11.4	0.329	0.0178	0.150	−0.893	−0.141	0.0343	−0.328	0.0643	0.367
16	−0.0186	−0.443	1.82	0.482	−0.270	0.625	2.24	0.00840	1.46	−0.374	−0.623	−0.565	0.386	−0.442	0.492	1.81	−0.702	2.24
17	3.13	−1.12	1.17	2.26	−0.483	1.04	2.07	−1.02	0.474	−0.674	−1.08	−2.85	0.0722	−0.451	−0.115	3.54	−0.238	2.07
18	0.109	0.402	−0.00585	1.36	0.291	1.54	3.25	0.649	1.05	−0.0941	1.65	1.53	−1.42	0.00184	0.207	1.15	1.75	3.25
19	−0.760	3.71	1.28	0.782	1.71	1.18	2.28	5.92	2.24	0.00342	−0.502	−0.289	−0.0729	−0.581	0.0233	2.04	−0.0688	2.28
20	−3.48	0.354	0.0403	−0.256	1.69	0.100	−13.7	−0.205	−3.13	−2.88	−1.01	−0.200	−0.653	−0.857	−0.550	0.334	−0.490	8.23
21	3.88	−0.136	17.5	−1.23	−2.09	−1.05	−3.62	−0.516	1.47	−0.797	0.605	0.185	−1.09	−0.00263	−2.02	−2.94	0.662	−0.722
22	1.38	1.92	−1.35	2.21	2.29	3.31	−3.45	0.198	−9.10	−5.80	−0.714	−1.04	0.257	1.45	−0.241	−9.40	−0.335	−2.11
23	0.447	0.0556	0.572	0.518	−0.155	0.411	0.877	0.165	2.19	−1.32	−5.69	2.65	−1.03	−1.96	−1.03	−2.27	−3.59	0.332
24	−0.433	0.354	0.256	−0.252	0.0145	0.279	0.968	0.367	1.30	−5.45	−1.12	−19.4	−0.271	2.02	−1.39	0.773	−0.165	−7.22
25	0.821	0.0267	−0.985	−0.143	−0.102	−0.614	2.03	0.229	−1.77	0.451	−0.888	0.738	−2.37	−0.800	−2.31	−1.23	−0.878	−3.83
26	−0.410	0.360	4.13	1.76	−0.551	2.71	1.50	0.244	11.8	−3.60	−0.178	0.413	−0.230	0.592	−0.515	−5.72	−0.202	0.564
27	2.59	0.753	−0.232	0.371	−0.361	0.0639	4.84	0.828	1.35	1.32	−0.365	2.55	−0.735	0.621	−1.01	−7.70	−0.961	−10.2

TABLE 2 *The wavelet coefficients of example 6.1 for* $s = 1/2$, *part B.* $d_{i,j,k,\kappa} := D_{i,j}(k,\kappa)$

k	$d_{1,0,k,1}$	$d_{1,0,k,2}$	$d_{1,0,k,3}$	$d_{1,0,k,4}$	$d_{1,0,k,5}$	$d_{1,0,k,6}$	$d_{1,0,k,7}$	$d_{1,0,k,8}$	$d_{1,0,k,9}$	$d_{1,1,k,1}$	$d_{1,1,k,2}$	$d_{1,0,k,3}$	$d_{1,1,k,4}$	$d_{1,1,k,5}$	$d_{1,1,k,6}$	$d_{1,1,k,7}$	$d_{1,1,k,8}$	$d_{1,1,k,9}$
1	2.66	0	0	0	0.143	0	0	1.16	0	14.7	0	0	0	−1.46	0	0	0	0
2	11.6	0	0	0	1.15	0	0	1.48	1.52	2.60	0	0	0	−0.278	0	0	0	0
3	−9.01	0	0	0	0.290	0	0	−0.903	0.845	4.54	2.20	0	0	0.203	0	0	0	0
4	−0.500	0	0	0	0.0383	0	0	0.132	−0.623	3.51	0.133	8.59	0	−0.958	0	0	0	0
5	5.16	1.63	0	0	−0.355	0	0	−0.0544	1.69	1.25	0.541	−3.34	1.63	0.742	0	0	0	0
6	−2.18	1.61	3.56	0	0.463	0	0	1.49	−1.51	−5.45	−0.280	−2.32	1.61	1.27	3.56	0	0	0
7	−2.08	−0.0587	−0.202	2.87	0.511	0	0	−1.41	0.917	1.12	−0.00154	−0.686	−0.0587	−0.187	−0.202	2.87	0	0
8	−0.172	−1.71	−0.882	1.95	0.665	3.08	0	−0.318	−1.10	−1.54	0.991	−3.58	−1.71	0.316	−0.882	1.95	3.08	0
9	−2.09	0.240	−2.42	0.0495	0.671	0.326	4.14	2.05	−0.849	−2.68	−0.500	−1.89	0.240	−0.257	−2.42	0.0495	0.326	4.14
10	−2.82	1.08	0.0230	−0.920	0.673	0.0236	−2.06	2.49	2.26	−0.818	−1.80	−0.0307	1.08	0.174	0.0230	−0.920	0.0236	−2.06
11	−2.02	−1.28	−0.846	−0.352	−1.38	−0.230	1.47	−2.06	9.18	−3.52	−0.441	0.247	−1.28	−0.415	−0.846	−0.352	−0.230	1.47
12	1.46	0.529	0.213	−0.948	−0.366	0.846	−0.488	−1.17	−3.46	2.79	3.59	0.132	0.529	1.02	0.213	−0.948	0.846	−0.488
13	4.80	−0.374	0.170	0.457	−0.180	−0.974	1.29	−1.44	−10.0	4.32	−0.882	11.2	−0.374	−1.67	0.170	0.457	−0.974	1.29
14	1.01	−2.41	3.05	0.138	−1.16	0.330	0.297	−2.80	−0.370	1.30	−0.456	−2.78	5.71	1.64	3.05	0.138	0.330	0.297
15	−3.72	−0.0537	−11.4	0.0178	1.92	0.150	−0.893	−1.22	−8.79	−1.52	−0.781	−3.33	1.29	0.810	1.49	0.0178	0.150	−0.893
16	1.93	0.00840	1.46	−0.623	−0.322	−0.565	0.386	0.0146	−2.55	3.70	−1.12	−8.19	−0.0357	−1.50	−0.917	17.5	−0.565	0.386
17	3.45	−1.02	0.474	−1.08	−0.539	−2.85	0.0722	−0.0327	−5.38	−3.99	4.46	−1.77	0.626	2.26	1.02	−1.94	3.81	0.0722
18	2.71	0.649	1.05	1.65	0.901	1.53	−1.42	1.11	−3.22	−5.34	−0.961	1.02	−0.791	−1.62	−0.300	0.681	−1.72	14.0
19	−0.332	−3.81	2.24	−0.502	−1.46	−0.289	−0.0729	−3.70	−0.612	−0.776	−0.434	−1.43	−0.744	0.0343	−0.541	−1.28	0.702	0.148
20	−3.20	−1.14	−3.13	−1.01	−0.0749	−0.200	−0.653	−0.538	2.26	1.95	1.29	5.19	1.29	−0.795	1.01	4.99	1.04	5.84
21	0.0810	−0.399	1.47	0.605	0.0301	0.185	−1.09	0.269	0.332	0.0380	0.0985	−0.263	0.128	0.235	0.362	−0.937	−0.507	−0.708
22	3.47	−0.274	6.16	−0.714	−1.40	−1.04	0.257	−0.200	1.70	−1.13	−0.969	−2.63	−0.668	0.0985	0.235	−1.39	−1.34	−1.57
23	0.552	0.919	−1.47	4.77	1.95	2.65	−1.03	0.763	−1.72	−0.113	−0.168	0.0585	−0.0643	−0.117	−0.185	0.346	0.0626	0.379
24	−0.920	0.234	−1.70	1.30	0.791	1.58	−0.271	0.473	0.749	0.425	0.242	0.524	0.344	−0.112	0.113	1.29	0.464	1.60
25	3.92	−0.464	−1.83	0.523	−2.01	−0.340	17.5	0.777	6.57	1.06	0.711	2.80	0.789	−0.343	0.968	2.68	0.645	3.47
26	−1.95	−0.409	−9.35	−1.92	0.171	−1.40	3.41	−0.917	−3.78	−1.93	−1.63	−2.69	−1.58	−0.299	−0.812	−2.31	−1.02	−0.558
27	−4.87	−1.22	−1.46	−2.49	0.0441	−1.59	−8.01	−1.18	0.437	2.23	1.62	5.67	1.55	−0.688	0.884	6.16	1.75	6.45



(a) ϕ^1 (b) ϕ^2 (c) ϕ^5

FIG. 3. Three of the scaling functions of the example presented in Section 6.1. The rest of the scaling functions are rotations of ϕ^1 or ϕ^2.

where

$$\mathbb{A}_0 = \{0, 1, \ldots, N_1\} \times \cdots \times \{0, 1, \ldots, N_d\},$$

$$\mathbb{A}_1 = \{1, 2, \ldots, N_1\} \times \cdots \times \{1, 2, \ldots, N_d\},$$

thus we may rewrite Δ as follows: $\Delta = \{(x_{\boldsymbol{i}}, z_{\boldsymbol{i}}) \in I \times \mathbb{R}, \boldsymbol{i} \in \mathbb{A}_0\}$. Let $\{\boldsymbol{e}_{d,l}; l = 1, \ldots, d\}$ be the standard basis of $\mathbb{R}^d$. Furthermore, for any $x \in \mathbb{R}^d$, $x = (x_1, \ldots, x_d)$, we use the notations $\mathrm{proj}_{-\lambda}\boldsymbol{x}$ and $\mathrm{proj}_{\lambda}\boldsymbol{x}$ as follows:

$$\mathrm{proj}_{-\lambda}\boldsymbol{x} = (x_1, \ldots, x_{\lambda-1}, x_{\lambda+1}, \ldots, x_d) \in \mathbb{R}^{d-1},$$

$$\mathrm{proj}_{\lambda}\boldsymbol{x} = (x_1, \ldots, x_{\lambda-1}, 0, x_{\lambda+1}, \ldots, x_d) \in \mathbb{R}^{d}.$$

The interpolation points divide $[0, 1]^d$ into $\prod_{l=1}^{d} N_l$ regions:

$$I_{\boldsymbol{i}} = \left[x_{1,i_1-1}, x_{1,i_1}\right] \times \left[x_{2,i_2-1}, x_{2,i_2}\right] \times \cdots \times \left[x_{d,i_d-1}, x_{d,i_d}\right],$$

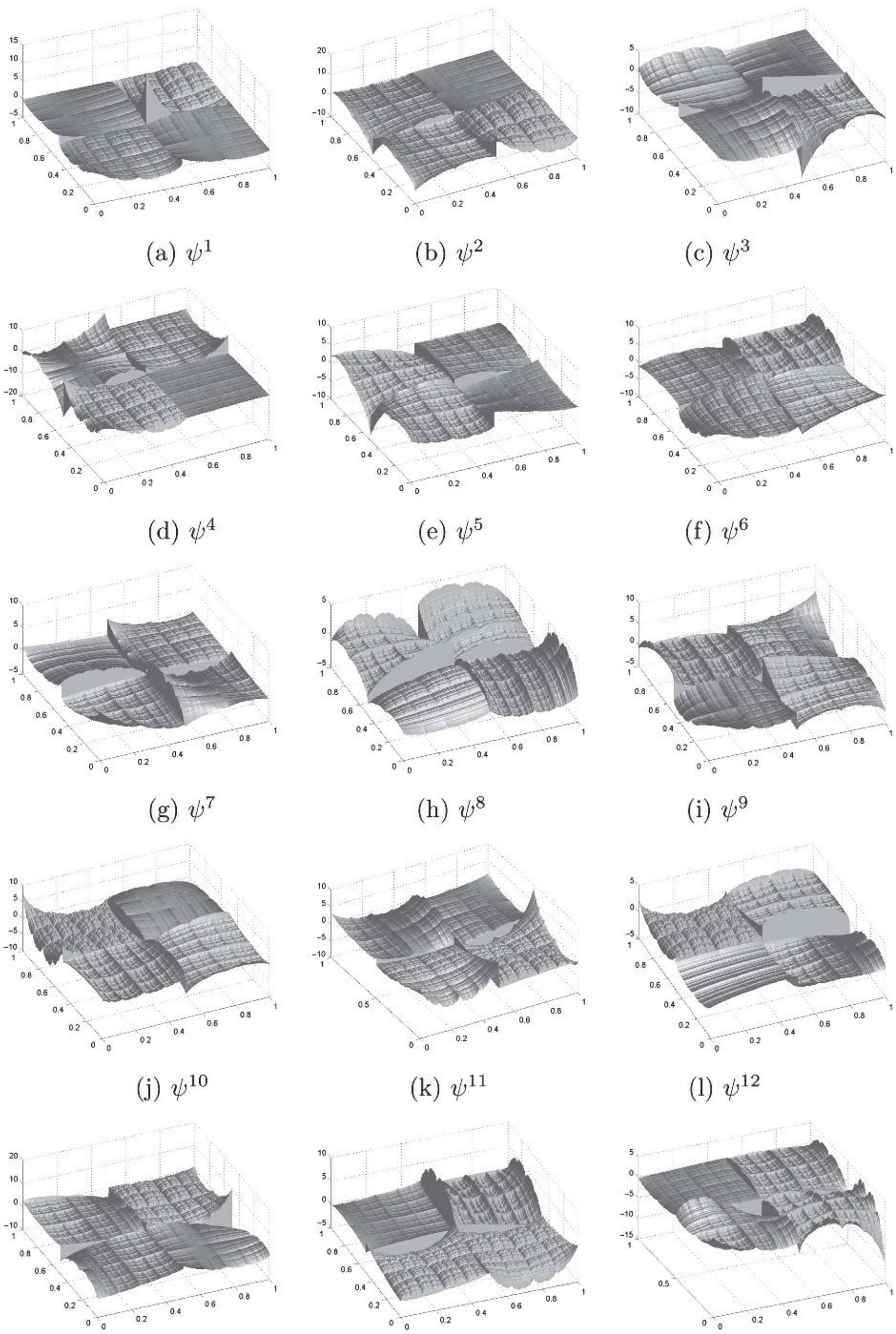
for all $\boldsymbol{i} \in \mathbb{A}_1$. Next we consider $\prod_{l=1}^{d} N_l$ mappings of the form

$$\boldsymbol{W_i}\colon [0, 1]^d \times \mathbb{R} \to I_{\boldsymbol{i}} \times \mathbb{R}\colon \boldsymbol{W_i}\begin{pmatrix} \boldsymbol{x} \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{T_i}(\boldsymbol{x}) \\ s_{\boldsymbol{i}} z + p_{\boldsymbol{i}}(\boldsymbol{x}) \end{pmatrix}, \tag{31}$$

$$\text{with} \quad \boldsymbol{T_i}(\boldsymbol{x}) = \begin{pmatrix} T_{1,i_1}(x_1) \\ T_{2,i_2}(x_2) \\ \vdots \\ T_{d,i_d}(x_d) \end{pmatrix},$$

for all $\boldsymbol{x} = (x_1, \ldots, x_d) \in [0, 1]^d$, where $p_{\boldsymbol{i}}$ is a continuous function that satisfies a Hölder condition of the form

$$|p_{\boldsymbol{i}}(\boldsymbol{x}) - p_{\boldsymbol{i}}(\boldsymbol{y})| \leqslant L_{\boldsymbol{i}} \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_1^{h_{\boldsymbol{i}}},$$



(a) ψ^1 (b) ψ^2 (c) ψ^3

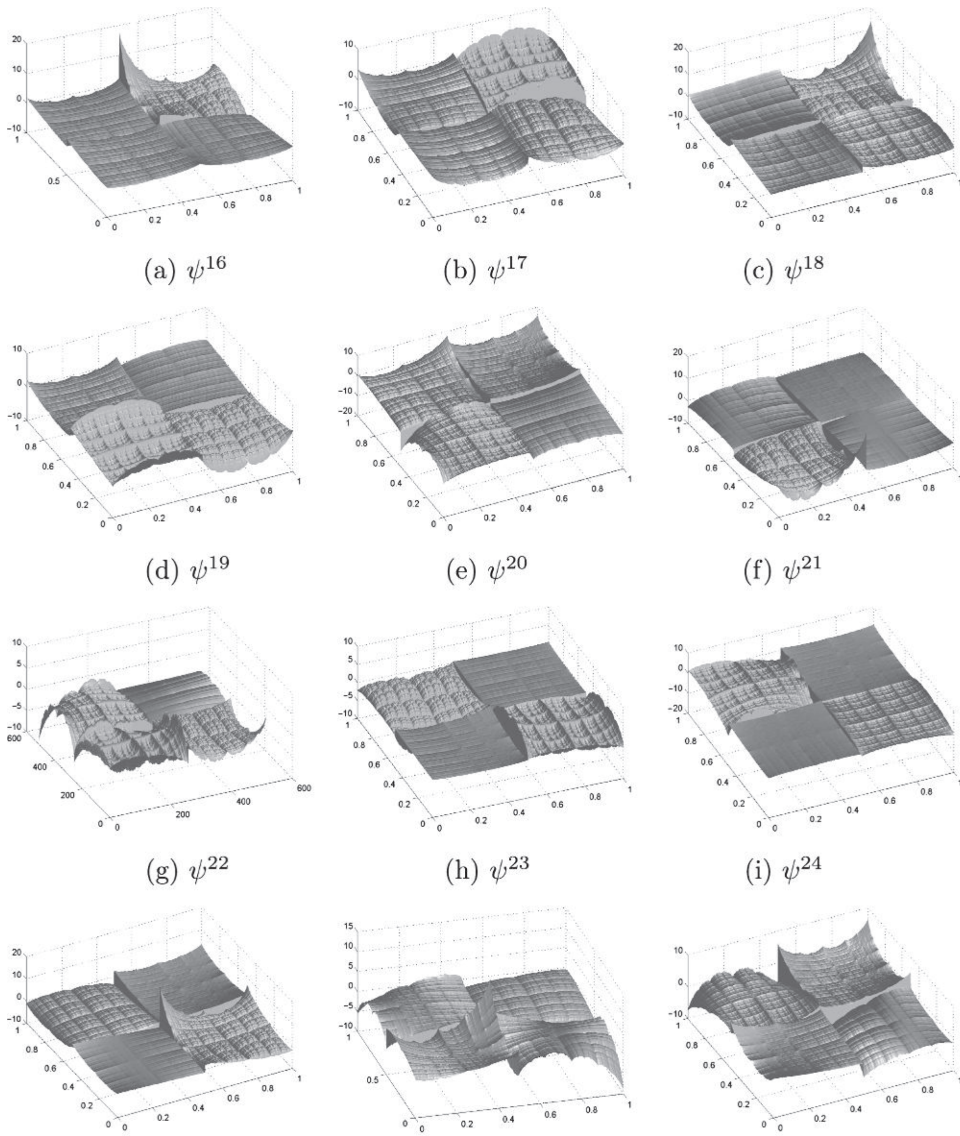
(d) ψ^4 (e) ψ^5 (f) ψ^6

(g) ψ^7 (h) ψ^8 (i) ψ^9

(j) ψ^{10} (k) ψ^{11} (l) ψ^{12}

(m) ψ^{13} (n) ψ^{14} (o) ψ^{15}

FIG. 4. The orthonormal multi-wavelets of example 6.1 for $s = 1/2$. Part A.



(a) ψ^{16} (b) ψ^{17} (c) ψ^{18}

(d) ψ^{19} (e) ψ^{20} (f) ψ^{21}

(g) ψ^{22} (h) ψ^{23} (i) ψ^{24}

(j) ψ^{25} (k) ψ^{26} (l) ψ^{27}

FIG. 5. The orthonormal multi-wavelets of example 6.1 for $s = 1/2$. Part B.

for some constants $L_{\boldsymbol{i}} > 0$, $h_{\boldsymbol{i}} > 0$, for all $\boldsymbol{i} \in \mathbb{A}_1$. Again, we confine the map $W_{\boldsymbol{i}}$ so that it maps the interpolation points with coordinates on the vertices of $[0, 1]^d$ to the interpolation points with coordinates on the vertices of $I_{\boldsymbol{i}}$, i.e.

$$T_{l,\boldsymbol{i}}(0) = x_{l,i_l-1}, \quad T_{l,\boldsymbol{i}}(1) = x_{l,i_l}$$

and

$$p_{\boldsymbol{i}}\left(\mathbf{1}-\sum_{\lambda=1}^{d}\delta_\lambda \boldsymbol{e}_{d,\lambda}\right)=z_{\boldsymbol{i}-\sum_{\lambda=1}^{d}\delta_\lambda \boldsymbol{e}_{d,\lambda}}-s_{\boldsymbol{i}}\cdot z_{\boldsymbol{N}-\sum_{\lambda=1}^{d}N_\lambda\delta_\lambda \boldsymbol{e}_{d,\lambda}}, \tag{32}$$

for all $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_d)\in\{0,1\}^d$, where $\boldsymbol{N}=(N_1,\ldots,N_d)$ and $\mathbf{1}=(1,\ldots,1)$.

As in Section 3, we can define a metric (equivalent to Euclidean metric) such that $W_{\boldsymbol{i}}$ is a contraction, $\boldsymbol{i}\in\mathbb{A}_1$. Thus, the resulting IFS $\{\mathbb{R}^{d+1}, \boldsymbol{W_i}; \boldsymbol{i}\in\mathbb{A}_1\}$ has a unique fixed point. The following proposition is a special case of a result found in Bouboulis & Dalla (2007a).

PROPOSITION 7.1 Let h be a continuous function that satisfies a Hölder property and interpolates the points of Δ. If the above-mentioned IFS satisfies the conditions

$$p_{\boldsymbol{i}}(\text{proj}_\lambda \boldsymbol{x}+\boldsymbol{e}_{d,\lambda})=h\left(\text{proj}_\lambda \boldsymbol{T_i}(\boldsymbol{x})+x_{\lambda,i_\lambda}\boldsymbol{e}_{d,\lambda}\right)-s_{\boldsymbol{i}}\cdot h(\text{proj}_\lambda \boldsymbol{x}+\boldsymbol{e}_{d,\lambda}), \tag{33}$$

$$p_{\boldsymbol{i}}(\text{proj}_\lambda \boldsymbol{x})=h\left(\text{proj}_\lambda \boldsymbol{T_i}(\boldsymbol{x})+x_{\lambda,i_\lambda-1}\boldsymbol{e}_{d,\lambda}\right)-s_{\boldsymbol{i}}\cdot h(\text{proj}_\lambda \boldsymbol{x}), \tag{34}$$

for all $x\in[0,1]^d$, $\boldsymbol{i}\in\mathbb{A}_1$, $\lambda=1,\ldots,d$ (where $s_{\boldsymbol{i}}$ are free parameters), then its attractor G is the graph of a continuous function f that also interpolates the data points.

We must point out that, as was the case when $d=2$ in Section 3, we are interested only with the values of h at the borders of $I_{\boldsymbol{i}}$. For the following, we assume that all $p_{\boldsymbol{i}}$ has the form

$$p_{\boldsymbol{i}}(\boldsymbol{x})=\sum_{l=1}^{d}r_{l,\boldsymbol{i}}(\text{proj}_{-l}\boldsymbol{x})\cdot x_l+\sum_{l=1}^{d}q_{l,\boldsymbol{i}}(\text{proj}_{-l}\boldsymbol{x}),$$

for all $\boldsymbol{i}\in\mathbb{A}_1$, that the vertical scaling factors are equal to s ($s_{\boldsymbol{i}}=s$, for all $\boldsymbol{i}\in\mathbb{A}_1$) and that the interpolation points are equidistant, i.e. Δ has the form

$$\Delta=\left\{\left(i_1/N, i_2/N,\ldots,i_d/N, z_{i_1,\ldots,i_d}\right); i_l=0,\ldots,N, l=1,\ldots,d\right\}.$$

Solving the system (33–34), we obtain the following relations:

$$\begin{aligned}r_{l,\boldsymbol{i}}(\text{proj}_{-l}\boldsymbol{x})=&\,h\left(\text{proj}_l\boldsymbol{T}(\boldsymbol{x})+x_{l,i_l}\boldsymbol{e}_{d,l}\right)-h\left(\text{proj}_l\boldsymbol{T}(\boldsymbol{x})+x_{l,i_l-1}\boldsymbol{e}_{d,l}\right)\\&-s_{\boldsymbol{i}}(h(\text{proj}_l\boldsymbol{x}+\boldsymbol{e}_{d,l})-h(\text{proj}_l\boldsymbol{x}))\\&-\sum_{k=1}^{l-1}(r_{k,\boldsymbol{i}}(\text{proj}_{-k,l}\boldsymbol{x}+\hat{\boldsymbol{e}}_{d-1,l})-r_{k,\boldsymbol{i}}(\text{proj}_{-k,l}\boldsymbol{x}+))\cdot x_k\\&-\sum_{k=1}^{l-1}(q_{k,\boldsymbol{i}}(\text{proj}_{-k,l}\boldsymbol{x}+\boldsymbol{e}_{d-1,l})-q_{k,\boldsymbol{i}}(\text{proj}_{-k,l}\boldsymbol{x}+))\end{aligned} \tag{35}$$

and

$$\begin{aligned}q_{l,\boldsymbol{i}}(\text{proj}_{-l}\boldsymbol{x})=&\,h(\text{proj}_l T_{\boldsymbol{i}}(\boldsymbol{x}))-s_i h(\text{proj}_l\boldsymbol{x})\\&-\sum_{k=1}^{l-1}r_{k,\boldsymbol{i}}(\text{proj}_{-k,l}\boldsymbol{x})\cdot x_k-r_{l,\boldsymbol{i}}(\text{proj}_{-l}\boldsymbol{x})\hat{x}_{l,j_l-1}-\sum_{k=1}^{l-1}q_{k,\boldsymbol{i}}(\text{proj}_{-k,l}\boldsymbol{x}),\end{aligned} \tag{36}$$

for all $\boldsymbol{x}=(x_1,\ldots,x_d)$, $\boldsymbol{i}\in\mathbb{A}_1$, $l=1,\ldots,d$.

As mentioned above, we will show that such an FIF satisfies a Hölder property. We will use this result to make a valid construction of FIFs that generate a multiresolution analysis. The search for the Holder exponents of FIFs is an issue addressed by many authors in the past. As an example, we mention the extensive work of Bedford (1989), where (among others) the relation between the Hölder exponent and the box-counting dimension of the affine FIF (with one variable) is explored, and the work of Tricot (1994). Results for the Hölder properties of a class of FIS can be found in Wang (2006). The proposition given below is a general result concerning FIFs defined on $[0, 1]^d$. We use the $\|\cdot\|_1$ norm to simplify the notation. Before we go on with the proof, we briefly describe some properties from the code spaces (or addresses) associated with the IFS (the interested reader can find more on the subject in Barnsley, 1993). A 'path' $\boldsymbol{j}_1, \boldsymbol{j}_2, \dots, \boldsymbol{j}_n, \dots$, where $\boldsymbol{j}_\lambda = (j_{1,\lambda}, j_{2,\lambda}, \dots, j_{d,\lambda})$, defines a sequence of mappings $\boldsymbol{W}_{\boldsymbol{j}_1}, \dots, \boldsymbol{W}_{\boldsymbol{j}_n}, \dots$ that are applied to an arbitrary point $(\boldsymbol{x}^{(0)}, z^{(0)})$ in the following way:

$$
\begin{aligned}
\boldsymbol{x}^{(1)} &= T_{\boldsymbol{j}_1}(\boldsymbol{x}^{(0)}), & z^{(1)} &= s z^{(0)} + p_{\boldsymbol{j}_1}(\boldsymbol{x}^{(0)}) \\
\boldsymbol{x}^{(2)} &= T_{\boldsymbol{j}_1} \circ T_{\boldsymbol{j}_2}(\boldsymbol{x}^{(0)}), & z^{(2)} &= s^2 z^{(0)} + s p_{\boldsymbol{j}_2}(\boldsymbol{x}^{(0)}) + p_{\boldsymbol{j}_1}\left(T_{\boldsymbol{j}_2}\left(T_{\boldsymbol{j}_2}(x^{(0)})\right)\right) \\
\boldsymbol{x}^{(3)} &= T_{\boldsymbol{j}_1} \circ T_{\boldsymbol{j}_2} \circ T_{\boldsymbol{j}_3}(x^{(0)}), & z^{(3)} &= s^3 z^{(0)} + s^2 p_{\boldsymbol{j}_3}(\boldsymbol{x}^{(0)}) + s p_{\boldsymbol{j}_2}\left(T_{\boldsymbol{j}3}(\boldsymbol{x}^{(0)}) + p_{\boldsymbol{j}_1}\left(T_{\boldsymbol{j}_2} \circ T_{\boldsymbol{j}_3}(\boldsymbol{x}^{(0)})\right)\right), \\
&\vdots & &\vdots
\end{aligned}
$$

At the nth iteration, we take

$$
\boldsymbol{x}^{(n)} = T_{\boldsymbol{j}_1} \circ T_{\boldsymbol{j}_2} \circ \cdots \circ T_{\boldsymbol{j}_n}(\boldsymbol{x}^{(0)}),
$$

$$
z^{(n)} = s^n z^{(0)} + s^{n-1} p_{\boldsymbol{j}_n}(x^{(0)}) + \sum_{r=1}^{n-1} s^{r-1} p_{\boldsymbol{j}_r}\left(T_{\boldsymbol{j}_{r+1}} \circ T_{\boldsymbol{j}_{r+2}} \circ \cdots \circ T_{\boldsymbol{j}_n}(x^{(0)})\right).
$$

This sequence converges to the point $(\boldsymbol{x}, f(x))$, $\boldsymbol{x} = (x_1, \dots, x_d)$, where $x_l = 0.j_{l,1} j_{l,2} \dots, j_{l,n}, \dots$ is a N-adic representation of x_l, $l = 1, \dots, d$, for any starting point $(\boldsymbol{x}^{(0)}, z^{(0)}) \in [0, 1]^d \times \mathbb{R}$.

PROPOSITION 7.2 (Hölder property of an FIF)

Let $f\colon [0, 1]^d \to \mathbb{R}$ be an FIF that interpolates the data set

$$
\Delta = \left\{\left(i_1/N, i_2/N, \dots, i_d/N, z_{i_1,\dots,i_d}\right); i_l = 0, \dots, N, l = 1, \dots, d\right\}.
$$

Let $\{\mathbb{R}^{d+1}, \boldsymbol{W}_{\boldsymbol{i}}; \boldsymbol{i} \in \mathbb{A}_1\}$ be the associated IFS as defined above. We assume that $p_{\boldsymbol{i}}$ satisfies a Hölder property of the form $|p_{\boldsymbol{i}}(\boldsymbol{x}) - p_{\boldsymbol{i}}(\boldsymbol{y})| \leqslant L_{\boldsymbol{i}} \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_1^{h_{\boldsymbol{i}}}$, where $L_{\boldsymbol{i}}, h_{\boldsymbol{i}} > 0$ for all $\boldsymbol{i} \in \mathbb{A}_1$. Then, there is a Hölder exponent $0 < h_0 \leqslant h = \min\{h_{\boldsymbol{i}}; \boldsymbol{i} \in \mathbb{A}_1\}$ and a positive number L_0 such that $|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leqslant L_0 \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_1^{h_0}$, for all $\boldsymbol{x}, \boldsymbol{y} \in [0, 1]^d$.

Proof. We define $h = \min\{h_{\boldsymbol{i}}, \boldsymbol{i} \in \mathbb{A}_1\}$, $L = \max\{L_{\boldsymbol{i}}, \boldsymbol{i} \in \mathbb{A}_1\}$ and $M = \max\{\|p_{\boldsymbol{i}}\|_\infty, \boldsymbol{i} \in \mathbb{A}_1\}$. Now, consider $\boldsymbol{x}, \boldsymbol{y} \in [0, 1]^d$ such that $\boldsymbol{x} = (x_1, x_2, \dots, x_d)$, $\boldsymbol{y} = (y_1, y_2, \dots, y_d)$, where

$$
\begin{aligned}
x_1 &= 0.j_{1,1} j_{1,2} \dots, j_{1,n}, \dots & y_1 &= 0.\hat{j}_{1,1} \hat{j}_{1,2} \dots, \hat{j}_{1,n}, \dots \\
x_2 &= 0.j_{2,1} j_{2,2} \dots, j_{2,n}, \dots & y_2 &= 0.\hat{j}_{2,1} \hat{j}_{2,2} \dots, \hat{j}_{2,n}, \dots \\
&\vdots & &\vdots \\
x_d &= 0.j_{d,1} j_{d,2} \dots, j_{d,n}, \dots & y_d &= 0.\hat{j}_{d,1} \hat{j}_{d,2} \dots, \hat{j}_{d,n}, \dots
\end{aligned}
$$

are the N-adic representations of $\boldsymbol{x}, \boldsymbol{y}$. If one of them is finite, then we fill the empty places with zeros. Thus, after applying the paths $\boldsymbol{j}_1, \boldsymbol{j}_2, \ldots, \boldsymbol{j}_n, \ldots$ and $\hat{\boldsymbol{j}}_1, \hat{\boldsymbol{j}}_2, \ldots, \hat{\boldsymbol{j}}_n, \ldots$ to any starting point $(\boldsymbol{x}^{(0)}, z^{(0)}) \in [0,1]^d \times \mathbb{R}$, we have

$$|z^{(n)} - \hat{z}^{(n)}| \leqslant |s|^n |z^{(0)} - \hat{z}^{(0)}| + |s|^{n-1} \left| p_{\boldsymbol{j}_n}(x^{(0)}) - p_{\hat{\boldsymbol{j}}_n}(x^{(0)}) \right| + \sum_{r=1}^{n-1} |s|^{r-1} \left| p_{\boldsymbol{j}_r}\left(T_{\boldsymbol{j}_{r+1}} \circ T_{\boldsymbol{j}_{r+2}} \circ \cdots \circ T_{\boldsymbol{j}_n}(x^{(0)})\right) - p_{\hat{\boldsymbol{j}}_r}\left(T_{\hat{\boldsymbol{j}}_{r+1}} \circ T_{\hat{\boldsymbol{j}}_{r+2}} \circ \cdots \circ T_{\hat{\boldsymbol{j}}_n}(x^{(0)})\right) \right|. \tag{37}$$

Evidently, there is an integer number $k \geqslant 1$ such that $\frac{1}{N^{k+1}} < \|x - y\|_1 \leqslant \frac{1}{N^k}$. We divide $[0,1]^d$ into hypercubes of side-length $1/N^k$.

Case I: There is a hypercube of side-length $1/N^k$ that contains both $\boldsymbol{x}$ and $\boldsymbol{y}$. This means that we can found N-adic representations of $\boldsymbol{x}, \boldsymbol{y}$, with the same first k digits. Assuming that this is the case in (37) and taking into account the Hölder property of $p_{\boldsymbol{i}}$ and the relation $|p_{\boldsymbol{i}}(\boldsymbol{x})| \leqslant M$, for all $\boldsymbol{i} \in \mathbb{A}_1$, we take

$$\begin{aligned} |z^{(n)} - \hat{z}^{(n)}| &\leqslant |s|^n |z^{(0)} - \hat{z}^{(0)}| + |s|^{n-1} \left| p_{\boldsymbol{j}_n}(x^{(0)}) - p_{\hat{\boldsymbol{j}}_n}(x^{(0)}) \right| \\ &\quad + \sum_{r=1}^{k} |s|^{r-1} \left| p_{\boldsymbol{j}_r}\left(T_{\boldsymbol{j}_{r+1}} \circ \cdots \circ T_{\boldsymbol{j}_n}(x^{(0)})\right) - p_{\hat{\boldsymbol{j}}_r}\left(T_{\boldsymbol{j}_{r+1}} \circ \cdots \circ T_{\boldsymbol{j}_k} \circ T_{\hat{\boldsymbol{j}}_{k+1}} \circ \cdots \circ T_{\hat{\boldsymbol{j}}_n}(x^{(0)})\right) \right| \\ &\quad + \sum_{r=k+1}^{k} |s|^{r-1} \left| p_{\boldsymbol{j}_r}\left(T_{\boldsymbol{j}_{r+1}} \circ \cdots \circ T_{\boldsymbol{j}_n}(x^{(0)})\right) - p_{\hat{\boldsymbol{j}}_r}\left(T_{\hat{\boldsymbol{j}}_{r+1}} \circ \cdots \circ T_{\hat{\boldsymbol{j}}_n}(x^{(0)})\right) \right| \\ &\leqslant |z^{(0)} - \hat{z}^{(0)}| + |s|^{n-1} 2M + \sum_{r=1}^{k} |s|^{r-1} L \left(\frac{1}{N^{k-r}}\right)^h + \sum_{r=k+1}^{n-1} |s|^{r-1} 2M. \end{aligned}$$

Taking limits as $n \to \infty$, we obtain

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leqslant L \cdot \sum_{r=1}^{k} |s|^{r-1} \left(\frac{1}{N^{k-r}}\right)^h + 2M \frac{|s|^k}{1 - |s|}.$$

Since $|s| < 1$, $N > 1$, we can choose $h_0 \leqslant h$ such that $|s| < (1/N)^{h_0} < 1$. Thus, if we define $M_2 = |s| N^{h_0} < 1$, after some algebra the above relation becomes

$$\begin{aligned} |f(\boldsymbol{x}) - f(\boldsymbol{y})| &\leqslant L \cdot \left(\sum_{r=1}^{k} M_2^{r-1}\right) \cdot \left(\frac{1}{N^{k-1}}\right)^{h_0} + \frac{2M \cdot M_2 \cdot N^{h_0}}{1 - |s|} \cdot \left(\frac{1}{N^{k+1}}\right)^{h_0} \\ &\leqslant \left(L \cdot M_3 \cdot N^{2h_0} + \frac{2M \cdot M_2 \cdot N^{h_0}}{1 - |s|}\right) \cdot \left(\frac{1}{N^{k+1}}\right)^{h_0}, \end{aligned}$$

where $M_3 = \sum_{r=0}^{\infty} M_2^r$. Since $\frac{1}{N^{k+1}} < \|\boldsymbol{x} - \boldsymbol{y}\|_1 \leqslant \frac{1}{N^k}$, we have finally that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leqslant M_4 \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_1^{h_0},$$

where

$$M_4 = \left(L \cdot M_3 \cdot N^{2h_0} + \frac{2M \cdot M_2 \cdot N^{h_0}}{1-|s|}\right).$$

Since h_0 and L do not depend on k, we have the result.

Case II: There is not a hypercube of side-length $1/N^k$ that contains both $\boldsymbol{x}$ and $\boldsymbol{y}$. Then there are two adjacent hypercubes of side-length $1/N^k$ such that the first contains $\boldsymbol{x}$ and the second contains $\boldsymbol{y}$. Let $\tilde{\boldsymbol{x}}$ be a common point of the two hypercubes (placed on their border), then we have

$$\begin{aligned}
|f(\boldsymbol{x}) - f(\boldsymbol{y})| &\leqslant |f(\boldsymbol{x}) - f(\tilde{\boldsymbol{x}})| + |f(\tilde{\boldsymbol{x}}) - f(\boldsymbol{y})| \\
&\leqslant M_4\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_1^{h_0} + M_4\|\boldsymbol{y} - \tilde{\boldsymbol{x}}\|_1^{h_0} \\
&\leqslant 2M_4\left(\frac{1}{N^k}\right)^{h_0} \\
&\leqslant 2M_4N^{h_0}\left(\frac{1}{N^{k+1}}\right)^{h_0} \\
&\leqslant 2M_4N^{h_0} \cdot \|\boldsymbol{x} - \boldsymbol{y}\|_1^{h_0}.
\end{aligned}$$

This completes the proof. □

REMARK 7.1.

- It is evident that the Hölder exponent $h_0 \leqslant h$ computed above, such that $|s| < (1/N)^{h_0} < 1$, depends on $|s|$. If $|s|$ is relatively close to 1, then h_0 will tend to be close to 0. On the other hand, if $|s|$ is relatively small (i.e. $|s| < (1/N)^h$) then h_0 will be equal to h. This agrees with the results found in Bedford (1989) for the affine 1D FIFs.
- The result holds even if we consider different vertical scaling factors assigned to each $\boldsymbol{i} \in \mathbb{A}_1$.

7.2 *Multiresolution analysis of $L_2(\mathbb{R}^d)$*

Now we proceed to the point of making the construction of FIFs to depend only on the interpolation points (and of the free parameter s). This is done by induction. We demonstrated in Section 3 the procedure for $d = 2$. The resulting FIFs were called generalized-affine FISs. Now we assume that the construction of generalized affine FIFs is valid on $[0, 1]^{d-1}$ and we construct a generalized-affine FIF on $[0, 1]^d$. To this end, consider the data sets

$$\Delta_{l,j_l} = \left\{\left(\frac{j_1}{N}, \ldots, \frac{j_{l-1}}{N}, \frac{j_{l+1}}{N}, \ldots, \frac{j_d}{N}, z_{j_1,\ldots,j_d}\right);\ j_k = 0, \ldots, N, k = 1, \ldots, d, k \neq l\right\},$$

for all $j_l = 0, \ldots, N$, $l = 1, \ldots, d$. Let $\upsilon_{l,j_l}\colon [0, 1]^{d-1} \to \mathbb{R}$, be the generalized-affine FIF associated with Δ_{l,j_l} and s, for all $j_l = 0, \ldots, N$, $l = 1, \ldots, d$. Using relations (32), (33) and (34), we can easily prove that $p_{\boldsymbol{i}}$ will be given in terms of υ_{l,j_l} for all $j_l = 0, \ldots, N$, $l = 1, \ldots, d$ and for all $\boldsymbol{i} \in \mathbb{A}_1$. In fact, $p_{\boldsymbol{i}}$ will be given as a linear combination of υ_{l,j_l}, some polynomials and υ_{l,j_l} multiplied with combinations of the coordinates of $\boldsymbol{x}$ (see, e.g. (22)). By induction, Proposition 7.2 ensures that $p_{\boldsymbol{i}}$ will satisfy a Hölder property for all $\boldsymbol{i} \in \mathbb{A}_1$ making this construction valid. In addition, we can easily see that certain properties such as the ones presented in Section 5 also hold for $d > 2$.

Finally, consider the $(N+1)^d$ generalized-affine FIFs each one interpolating the set of points

$$\Delta_{k_1,\ldots,k_d} = \left\{ \left(\frac{i_1}{N}, \ldots, \frac{i_d}{N}, z_{i_1,\ldots,i_d} \right); z_{k_1,\ldots,k_d} = 1, z_{i_1,\ldots,i_d} = 0, \text{ for all other indices} \right\},$$

for all $(k_1, \ldots, k_d) \in \mathbb{A}_0$. Using arguments similar to those in Sections 4, 5 and 6, we can prove that $\phi^1, \ldots, \phi^r$, where $r = (N+1)^d$, generate a multiresolution analysis of $L_2(\mathbb{R}^d)$. We can also compute the $(N+1)^d \cdot (N^d - 1)$ associated orthogonal wavelets. To avoid repeating the same arguments over and over, we omit the proofs.

8. Conclusions and future research

The new construction of FIFs on $[0,1]^d$ presented in Sections 3 and 7 generalizes the concept of affine FIFs defined on [0, 1]. We have shown that they can be effectively used for the generation of multiresolution analyses of $L_2(\mathbb{R}^d)$. The associated multi-wavelets are orthonormal, but discontinuous. The free parameter s may be used to involve certain constrains in the analyses. It is motivating to see if we can use the aforementioned construction to generate continuous multi-wavelets somewhere in the lines of Donovan *et al.* (1996).

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